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Hydrodynamic Limit for Particle Systems with Nonconstant Speed Parameter

Paul Covert[1] and Fraydoun Rezakhanlou[1]

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We establish the hydrodynamic limit for a class of particle systems on $\mathbb{Z}^d$ with nonconstant speed parameter, assuming that the speed parameter is continuously differentiable in the spatial variable. If the particle system is on the one-dimensional lattice $\mathbb{Z}$ and totally asymmetric, we derive the hydrodynamic equation for continuous speed parameters. We obtain nontrivial upper and lower bounds when either the speed parameter is discontinuous or there is a blockage at a fixed site.

KEY WORDS: Hydrodynamic limit; exclusion process; scalar conservation law.

1. INTRODUCTION

Recent studies by Janowsky and Lebowitz [JL1, JL2] have focused on the Totally Asymmetric Simple Exclusion Process (TASEP) with a blockage at one site. This is a one-dimensional lattice particle system in which particles may only jump one site in one direction, with a rate that is constant, except for one site at which it is slowed. This may be used as a model for such diverse physical phenomena as fluid motion in a pipe with a valve, traffic flow on a highway, and superionic conductors.

In [JL2], Janowsky and Lebowitz use computational methods to examine the question: "For a given initial particle density ρ, how slow may the jump rate at the blockage be without disturbing the hydrodynamics of the system?" They compute a numerical approximation to a function describing this relationship.

[1] Department of Mathematics, University of California, Berkeley, California 94720-3840.

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In this paper, we approach that question by means of smooth approximations to the discontinuous jump rate function of the system with a blockage. To this end, we study the hydrodynamic behavior of a certain type of stochastic particle system, the "Processus des Misanthropes" (translated literally "Process of Misanthropes," named with reference to the fact that particles in a process tend to space themselves out instead of piling up at a site), which includes the simple exclusion and zero-range processes as special cases. These systems consist of particles moving on a multi-dimensional lattice according to a Markovian law. Under Euler scaling, the microscopic particle density converges to a deterministic limit that is characterized as the solution of a nonlinear conservation law.

This paper expands on a result of Rezakhanlou [R], who established the hydrodynamic behavior for these and similar processes under the condition that the jump rate between sites depends only on the distance between the sites and not on the absolute macroscopic spatial coordinates of the site of origin. In this paper, that assumption is dropped, allowing jump rates to depend on a continuously differentiable, bounded, and uniformly positive function of the form $\lambda(u/N)$, where N is the Euler scaling factor.

The simplest process we will study, the simple exclusion process (hereafter called the SEP), is defined as follows: Let E denote the space of configurations $\eta=(\eta(u)\colon u\in\mathbb{Z}^d)$ when $\eta(u)$ may be either 1 or 0, corresponding to the presence or absence of a particle at site u. Let $p=(p(z)\colon z\in\mathbb{Z}^d)$ be a probability transition function (that is, $p(\cdot)$ takes only non-negative values and sums to 1). The generator $\mathscr{L}^N$ of an SEP with scaling factor N may then be wrritten in the form

$$\mathscr{L}^N f(\eta)=\sum_{u,v} p(v-u)\,\lambda\left(\frac{u}{N}\right)\eta(u)(1-\eta(v))(f(\eta^{u,v})-f(\eta)) \qquad (1.1)$$

for local functions f defined on the configuration space E, where $\eta^{u,v}$ denotes the configuration η with a particle moved from u to v if possible:

$$\begin{aligned}
\eta^{u,v}(u)&=\eta(u)-1 && \text{if}\quad \eta(u)=1,\quad \eta(v)=0\\
\eta^{u,v}(v)&=\eta(v)+1 && \text{if}\quad \eta(u)=1,\quad \eta(v)=0 \qquad (1.2)\\
\eta^{u,v}(z)&=\eta(z) && \text{if}\quad z\neq u,v
\end{aligned}$$

We shall also have occasion to use the generator $\mathscr{L}^{(1)}$ defined by replacing the function λ in (1.1) by the constant 1. Note that this generator is independent of the scaling factor N.

The construction of this process in the case $\lambda\equiv 1$ is described by Liggett in [Li].

The zero-range process (ZRP) is similar to this, except that the limitation of at most one particle per site is removed. The jump rate is then defined to depend on a nondecreasing function of the number of particles at the site of origin, as follows: Let $G: \mathbf{N} \to [0, \infty)$ be a bounded nondecreasing function with $G(0) = 0$, $G(1) = 1$. In this case, $\eta^{u,v}$ still refers to the configuration obtained by jumping a particle from u to v if possible, but the condition $\eta(u) = 1$, $\eta(v) = 0$ is replaced by only $\eta(u) > 0$ in both places where it occured in (1.2). A ZRP may then be defined as one with generator in the form:

$$\mathscr{L}^N f(\eta) = \sum_{u,v} p(v-u)\, \lambda\left(\frac{u}{N}\right) G(\eta(u))(f(\eta^{u,v}) - f(\eta))$$

The basic reference for this process is Andjel [A].

The general Processus des Misanthropes (PdM) will be discussed in detail in Section 2.

It is known that for any constant ρ in the possible range of densities for these processes, in the case of constant speed parameter, there is a unique translation-invariant equilibrium measure ν^ρ with density ρ. This ν^ρ will be a probability measure on the space of possible configurations, with the following properties:

$$\int \mathscr{L}^{(1)} f \, d\nu^\rho = 0$$

for all local functions f, that is, functions on the configuration space whose value depends only on finitely many coordinates;

$$\int \eta(0)\, \nu^\rho(d\eta) = \rho$$

$$\int \tau_u f \, d\nu^\rho = \int f \, d\nu^\rho \qquad \text{for all} \quad u \in \mathbb{Z}^d$$

where τ_u is the shift operator defined by

$$\tau_u f(\eta) = f(\tau_u \eta)$$

$$\tau_u \eta(v) = \eta(u+v) \qquad \text{for} \quad u, v \in \mathbb{Z}^d, \quad \eta \in E$$

In the simple exclusion case, ν^ρ is defined to be the product measure with probability ρ of a particle at any given site. For the general PdM, with the ZRP as a special case of it, see Section 2.

Because we will be using Euler scaling, we will consider the speeded generator $N\mathscr{L}^N$ for positive integers N. We will use η_t to denote a configuration on which this speeded generator has acted for time t, as there will in general be no possibility of confusion arising from the scaling factor.

The object of this paper is to derive the hydrodynamic equation for the macroscopic particle densities in the limit as $N \to \infty$. We define *density profiles* as follows: we say that a sequence μ^N of probability measures on the configuration space has density profile ρ, written

$$\mu^N \sim \rho$$

if

$$\lim_{N \to \infty} \int \left| \frac{1}{N^d} \sum_u J\left(\frac{u}{N}\right) \eta(u) - \int J(x)\, \rho(x)\, dx \right| \mu_0^N(d\eta) = 0$$

for all test functions J.

We will be starting with a distribution μ_0^N for the initial configuration η_0, with its density profile being some bounded measurable function ρ_0 on $\mathbb{R}^d$. We assume that the measure μ_0^N is a product measure such that

$$v_0^N(\eta(u) = n) = v^{\rho_{u,N}}(\eta(u) = n)$$

with the sequence $\rho_{u,N}$ satisfying

$$\lim_{N \to \infty} \int_{|x| \leqslant k} |\rho_{[Nx],N} - \rho_0(x)|\, dx = 0$$

for every k. Here $[Nx]$ denotes the integer part of $[Nx]$. Note that if ρ_0 is continuous function, we choose $\rho_{u,N} = \rho_0(u/N)$. Our goal will be to prove that the distribution of η_t for later t has a density profile $\rho(t, \cdot)$, where ρ satisfies the conservation law PDE

$$\partial_t \rho + \gamma \cdot \nabla_x \{\lambda(x)\, h(\rho(t, x))\} = 0 \tag{1.3}$$

for $x \in \mathbb{R}^d$ and $t \geqslant 0$, where $\gamma = \sum_z z p(z)$ and $h(\rho)$ is the expected flux for the equilibrium measure with density ρ. This is $\rho(1 - \rho)$ for the SEP; the definition of h for the general PdM will be given in Section 2.

The PDE (1.3) is here understood in its distributional sense. Since distribution solutions are not in general unique, we choose the appropriate solutions by means of the entropy condition

$$\partial_t |\rho - c| + \gamma \cdot \nabla_x \{\lambda q(\rho; c)\} + \gamma \cdot \nabla \lambda(x)\, h(c)\, \mathrm{sgn}(\rho - c) \leqslant 0 \tag{1.4}$$

for all $c \in \mathbb{R}$, where $q(\rho; c) = \operatorname{sgn}(\rho - c)(h(\rho) - h(c))$. The inequalities (1.4) are likewise understood distributionally. Kružkov's uniqueness theorem [K] assures us of a unique solution of (1.3) satisfying (1.4), provided that

$$\lim_{t \to 0} \int_{|x| \leqslant k} |\rho(x, t) - \rho_0(x)| \, dx = 0$$

for all constants k.

The work in this paper makes use of certain assumptions on the transition probability function p:

Assumptions 1.1. (a) $p(\cdot)$ is of finite range;

(b) $p(z)$ is irreducible.

This irreducibility is necessary to make use of ergodicity and monotonicity arguments using coupled pairs of configurations evolving together.

We are now ready to state our main result:

Theorem 1.2. Let E^N denote the expectation of the process η_t for $t \in [0, \infty)$ with η_0 distributed according to μ_0^N. Then, for any $t > 0$, smooth J of compact support, and $\delta > 0$,

$$\lim_{N \to \infty} E^N \left| \frac{1}{N^d} \sum_u J\left(\frac{u}{N}\right) \eta_t(u) - \int J(x) \, \rho(t, x) \, dx \right| = 0$$

where $\rho(t, x)$ is the unique solution of (1.3) satisfying the entropy inequalities (1.4) and $\rho(0, x) = \rho_0(x)$.

It turns out that in the case of zero-range process, certain product measures are in the invariant form $\mathscr{L}^N$. Because of this, the method of [R] can be used directly to establish Theorem 1.2. Even when there is a blockage at a site, some suitable product measures are invariant. This was used by Landim [La] to establish the hydrodynamic limit for such models. However in general the invariant measures are not product measures and because of this the arguments of [R] are not directly applicable. The key ideas in the proof are the following:

(1) By means of relative entropy techniques, we can show that if there is a smooth solution m of the PDE (1.4) for time $t \in [0, T]$ with $m(0, x) \equiv \rho_0(x)$, then the particle densities at later times actually do match the values of m. This is shown in Section 4.

(2) By arguments following those of Rezakhanlou in [R], we can couple our process η_t to a process ζ_t with smooth density profile m in a given time interval, and show that the particle densities of η and ζ satisfy

a microscopic version of the entropy inequalities, with the smooth functions m replacing the constants c in (1.4). This is done in Section 5.

(3) By means of a pair of PDE lemmas, we can show that this version of the entropy inequality is sufficient to prove that the original entropy inequality (1.4) (using the constants c) also holds. This is done in Section 6, the arguments of which are modeled after those of Kružkov in [K] and of DiPerna in [D].

Independently of us, Bahadoran also proves Theorem 1.2 in the case of ASEP. As the first step, he establishes the hydrodynamic limit for the invariant measures. He then replaces the constant c in (1.4) with a steady solution of the PDE.

Section 3 proves a preparatory lemma, a "one-block estimate," which will be used in Section 4. Section 7 concludes the paper by applying the result to obtain a partial answer to the question posed by Janowsky and Lebowitz for the TASEP in [JL2].

2. THE PROCESSUS DES MISANTHROPES

The simple exclusion and zero-range processes, described in Section 1, are special cases of a more general class of processes known as Processus des Misanthropes (hereafter called "PdM"), originally discussed by Cocozza [C]. The difference between the ZRP and the general PdM is that the ZRP allows jump rates between sites to depend on the number of particles only at the jumping-off site, while the PdM allows the rate to depend on the number of particles at the destination also.

Specifically, the generator of a PdM is of the following form:

$$\mathscr{L}^N f(\eta) = \sum_{u,v} p(v-u)\, \lambda\left(\frac{u}{N}\right) b(\eta(u), \eta(v))(f(\eta^{u,v}) - f(\eta))$$

where b is bounded, nondecreasing in its first variable, and nonincreasing in its second, and has the following properties:

$$b(0, m) = 0 \qquad \text{for all } m$$

$$b(n, m) - b(m, n) = b(n, 0) - b(m, 0); \text{ (Gradient Condition)}$$

$$b(n, m-1)\, G(m) = b(m, n-1)\, G(n) \text{ (Detailed Balance Condition)}$$

where G s a bounded nondecreasing function on $\mathbf{N}$ having $G(0) = 0$, $G(1) = 1$.

To construct a SEP, we require that the initial configuration have at most one particle per site, and that $b(1, 1) = 0$. In this case, $G(n)$ is undefined for $n > 1$, but the undefined values will not be needed in the work that follows.

To study the PdM, we will use several pieces of notation.

Let

$$\tilde{G}(n) = \prod_{i=1}^{n} G(i)$$

for $\alpha \in [0, \sup G)$, let

$$Z(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^n}{\tilde{G}(n)}$$

let

$$\rho(\alpha) = \sum_{n=0}^{\infty} \frac{n\alpha^n}{\tilde{G}(n)}$$

Note that ρ is a continuous, strictly increasing function of α. Let $\alpha(\rho)$ be the inverse function of ρ as defined above; let

$$\beta(\rho) = 1/Z(\alpha(\rho))$$

let

$$\Theta^{\rho}(n) = \frac{\beta(\rho)(\alpha(\rho))^n}{\tilde{G}(n)}$$

let

$$v^{\rho}(\eta) = \prod_{u} \Theta^{\rho}(\eta(u))$$

then the product measure v^{ρ} is invariant under the PdM generator if $\lambda \equiv 1$ identically.

Let

$$h(\rho) = \int b(\eta(u), \eta(v))\, v^{\rho}(d\eta)$$

for $u \neq v$; let

$$\hat{h}(\rho) = \int b(\eta(u), 0)\, v^{\rho}(d\eta)$$

(For the ZRP, $h(\rho) = \hat{h}(\rho)$.)

We have:

Lemma 2.1.

$$\frac{\beta'(\rho)}{\beta(\rho)} = -\rho\,\frac{\alpha'(\rho)}{\alpha(\rho)}$$

Proof. Note that

$$1 = \sum_{n=0}^{\infty} \frac{\beta(\rho)(\alpha(\rho))^n}{\tilde{G}(n)}$$

hence

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} \frac{d}{d\rho}\frac{\beta(\rho)(\alpha(\rho))^n}{\tilde{G}(n)} \\
&= \sum_{n=0}^{\infty} \frac{\beta(\rho)(\alpha(\rho))^n}{\tilde{G}(n)}\left[\frac{\beta'(\rho)}{\beta(\rho)} + n\,\frac{\alpha'(\rho)}{\alpha(\rho)}\right] \\
&= \frac{\beta'(\rho)}{\beta(\rho)} + \rho\,\frac{\alpha'(\rho)}{\alpha(\rho)}
\end{aligned}$$

the result follows. ∎

We also have:

Lemma 2.2. $\hat{h}'(\rho) = h(\rho)(\alpha'(\rho)/\alpha(\rho))$.

Proof.

$$h(\rho) = \sum_{n,m} \Theta^{\rho}(n)\,\Theta^{\rho}(m)\,b(n, m)$$

if $n \geqslant 1$, then

$$\begin{aligned}
&\Theta^{\rho}(n)\,\Theta^{\rho}(m)\,b(n, m) \\
&\quad = \frac{(\beta(\rho))^2(\alpha(\rho))^{n+1}}{\tilde{G}(n)\,\tilde{G}(m)}\,b(n, m) \\
&\quad = \frac{(\beta(\rho))^2(\alpha(\rho))^{n+m}\,b(m+1, n-1)}{\tilde{G}(n-1)\,\tilde{G}(m+1)} \\
&\quad = \Theta^{\rho}(n-1)\,\Theta^{\rho}(m+1)[b(n-1, m+1) - b(n-1, 0) + b(m+1, 0)]
\end{aligned}$$

where for the second and third line we used the detailed balance condition and the gradient condition respectively. Since $\Theta^{\rho}(n)\,\Theta^{\rho}(m)\,b(n,m)$ is equal to

$$\Theta^{\rho}(n-1)\,\Theta^{\rho}(m+1)[b(n-1,m+1)-b(n-1,0)+b(m+1,0)]$$

it also equals

$$\Theta^{\rho}(n-2)\,\Theta^{\rho}(m+2)[b(n-2,m+2)-b(n-2,0)+b(m+2,0)]$$
$$+\Theta^{\rho}(n-1)\,\Theta^{\rho}(m+1)[-b(n-1,0)+b(m+1,0)]$$

whenever $n\geqslant 2$. Continuing in this manner, we eventually conclude that it equals

$$\sum_{k=0}^{n}\Theta^{\rho}(n-k)\,\Theta^{\rho}(m+k)[b(m+k,0)-b(n-k,0)]$$

In this case,

$$\begin{aligned}h(\rho)&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\Theta^{\rho}(n-k)\,\Theta^{\rho}(m+k)[b(m+k,0)-b(n-k,0)]\\&=\sum_{i,j}\Theta^{\rho}(i)\,\Theta^{\rho}(j)\,i[b(i,0)-b(j,0)]\\&=\sum_{i}\Theta^{\rho}(i)\,b(i,0)(i-\rho)\end{aligned}$$

Furthermore,

$$\begin{aligned}\hat{h}'(\rho)&=\frac{d}{d\rho}\sum_{i}[b(i,0)\,\Theta^{\rho}(i)]\\&=\sum_{i}b(i,0)\,\frac{\beta(\rho)(\alpha(\rho))^{i}}{\tilde{G}(i)}\left[\frac{\beta'(\rho)}{\beta(\rho)}+i\,\frac{\alpha'(\rho)}{\alpha(\rho)}\right]\\&=\sum_{i}b(i,0)\,\Theta^{\rho}(i)(i-\rho)\,\frac{\alpha'(\rho)}{\alpha(\rho)}\end{aligned}$$

The result follows. ▌

We will also need a result concerning the Large Deviations rate function of the invariant measures ν^{m} from the PdM with $\lambda\equiv 1$ and constant density m.

In the sequel, E_x will denote the expectation with respect to θ^x. For m, x in the range of possible densities, let ϕ be the microscopic entropy function

$$\phi(x, m) = \sup_y [xy - \Lambda_m(y)]$$

with Λ the logarithmic moment generating function

$$\Lambda_m(y) = \log E_m \exp ny$$

In the following lemma we give an exponential bound for the probability of a block average of η taking a given value. Let the block $T_\ell = \{-\ell, ..., \ell - 1\}^d$, and let the block average $M_{T_\ell}\eta = 1/|T_\ell| \sum_{u \in T_\ell} \eta(u)$ where $|T_\ell| = (2\ell)^d$. Then we have:

Lemma 2.3.

$$\nu^m\left(M_{T_\ell}\eta = \frac{k}{|T_\ell|}\right) \leqslant E_m \exp\left[-|T_\ell|\, \phi\left(\frac{n}{|T_\ell|}, m\right)\right]$$

We omit the standard proof of the above lemma. See for example [DZ, Chapter 2].

3. ONE-BLOCK ESTIMATE

In this section, we prove a result that will be used in Section 4. It allows us to replace the average of a local function over a large microscopic block with a corresponding function of the average of the configuration over that block. This is done with blocks of size ℓ in systems with scaling factor N, where N is sent to infinity first while ℓ is held constant (making the speed function λ approximately constant over blocks of size ℓ). Then ℓ is sent to infinity after that.

Lemma 3.1. Let $S_t^N = e^{tN\mathscr{L}^N}$ be the semigroup corresponding to the generator $N\mathscr{L}^N$, and write $\mu_t^N = S_t^N * \mu_0^N$, where μ_0^N is the initial measure on the configuration space. Let

$$\hat{\mu}^N = \frac{1}{N^d} \sum_{|u| \leqslant kN} \int_0^T \tau_{-u}\mu_t^N \, dt$$

Assume that μ^* is a limit point of $\{\hat{\mu}^N\}$. Then, for any local f,

$$\int \mathscr{L}^{(1)} f \, d\mu^* = 0$$

where $\mathscr{L}^{(1)}$ is the generator corresponding to the process with $\lambda \equiv 1$.

Proof.

$$\begin{aligned}
\int \mathscr{L}^N f\, d\hat{\mu}^N &= \frac{1}{N^d} \sum_{|u| \leqslant kN} \int_0^T \mathscr{L}^N f\, d(\tau_{-u} S_t^N * \mu_0^N)\, dt \\
&= \frac{1}{N^d} \sum_{|u| \leqslant kN} \int_0^T \int \tau_u S_y^N \mathscr{L}^N f\, d\mu_0^N\, dt \\
&= \frac{1}{N^d} \sum_{|u| \leqslant kN} \int_0^T \int \tau_u e^{tN\mathscr{L}^N} \mathscr{L}^N f\, d\mu_0^N\, dt \\
&= \frac{1}{N^{d+1}} \sum_{|u| \leqslant kN} \int_0^T \int \tau_u \frac{d}{dt} e^{tN\mathscr{L}^N} f\, d\mu_0^N\, dt \\
&= = \frac{1}{N^{d+1}} \sum_{|u| \leqslant kN} \int \tau_u [S_T^N f - f]\, d\mu_0^N = O\left(\frac{1}{N}\right) \quad \blacksquare
\end{aligned}$$

This formula tells us that $\lim_N \int \mathscr{L}^N f\, d\mu^* = 0$. But, since f is a local function, $\mathscr{L}^N f$ converges to a constant multiple of $\mathscr{L}^{(1)} f$ as N approaches infinity.

Lemma 3.2. For any local f and any j,

$$\int (f - \tau_j f)\, d\mu^* = 0$$

Proof.

$$\begin{aligned}
\left| \int (f - \tau_j f)\, d\hat{\mu}^N \right| &= \left| \frac{1}{N^d} \int_0^T \int \sum_{|u| \leqslant kN} (f - \tau_j f)\, \tau_{-u}\, d\mu_t^N\, dt \right| \\
&= \left| \frac{1}{N^d} \int_0^T \int \sum_{|u| \leqslant kN} (\tau_u f - \tau_{u+j} f)\, d\mu_t^N\, dt \right| \\
&= O\left(\frac{1}{N}\right)
\end{aligned}$$

The result follows. $\blacksquare$

Given this, a theorem of Cocozza [C] for the PdM tells us that

Corollary 3.3. For any local f,

$$\int f\, d\mu^* = \iint f\, d\nu^c\, \gamma(dc)$$

where γ is a measure on the space of possible densities and c ranges over that space.

Define the microscopic block $T_\ell(u)$ to be the set $\{u+v: v \in T_\ell\}$, and the block average $M_{T_\ell(u)}\eta$ to be $1/|T_\ell(u)| \sum_{z \in T(u)} \eta(z)$. Then we have:

Theorem 3.4. For any bounded local f, let $\hat{f}(c) = \int f \, d\nu^c$. Then, for any smooth J and almost all t,

$$\lim_{\ell \to \infty} \lim_{N \to \infty} \int_0^T \left| \frac{1}{N^d} \int \left| \frac{1}{N^d} \sum_u J\left(\frac{u}{N}\right) [M_{T_\ell(u)} f(\eta) - \hat{f}(M_{T_\ell(u)}\eta)] \right| d\mu_t^N \, dt = 0$$

Proof. By the Ergodic Theorem, $\int |M_{T_\ell} f(\eta) - \hat{f}(M_{T_\ell}\eta)| \, d\nu^c = 0$ in the limit as $\ell \to \infty$; the same holds for μ^*. Thus, as first N and then $\ell \to \infty$,

$$\frac{1}{N^d} \int_0^T \int \sum_u \left| J\left(\frac{u}{N}\right) \right| \, |M_{T_\ell} f(\eta) - \hat{f}(M_{T_\ell}\eta)| \, \tau_{-u} \, d\mu_t^N \, dt \to 0$$

the result follows. ∎

This is our one-block estimate.

4. RELATIVE ENTROPY ESTIMATE

In this section we use a relative entropy estimate to show that, so long as the expected density profile obtained from the PDE remains smooth, the particle densities do indeed converge to that profile. The use of a relative entropy estimate for the hydrodynamic limit was initiated in the articles Yau [Y] and Olla *et al.* [OYV].

We will be working here with configurations on the periodic lattice $\mathbb{Z}_N^d$ and with smooth functions on the d-dimensional torus $\mathbb{T}^d$; a natural coupling extends the result to configurations on $\mathbb{Z}^d$ and density profiles on $\mathbb{R}^d$. This will be discussed further at the end of this section.

Let η_t be generated by

$$N\mathscr{L}^N f(\eta) = N \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u) \, b(\eta(u), \eta(v))(f(\eta^{u,v}) - f(\eta))$$

having initial measure

$$\mu_0^N(\eta_0) = \Pi_u \Theta^{\rho_0}\left(\frac{u}{N}\right)(\eta(u))$$

where ρ_0 is a smooth function and Θ^{ρ_0} is as defined in Section 2. Let $\mu_t^N = S_t^N * \mu_0^N$, where S_t^N is the semigroup corresponding to $N\mathscr{L}^N$. (For the rest of this section, we suppress the N in the notation $\mathscr{L}^N$.)

Suppose the PDE (1.3) has a smooth solution m on $[0, T] \times \mathbb{T}^d$ with $\alpha(m(x, t)) > 0$ and $m(0, x) = \rho_0(x)$; let $v_t^N = \Pi_u \Theta^{m(t, u/N)}(\eta(u))$. Choose c; let $v^c = \Pi_u \Theta^c(\eta(u))$, $\psi_t = dv_t/dv^c$, and $f_t = d\mu_t/dv^c$. Let $\mathscr{L}^*$ be the adjoint of $\mathscr{L}$ with respect to v^c. (The function $f_t(\eta)$ solves the foward equation for the process.) Define the relative entropy H of a measure μ with respect to another measure v on the same space by $H(\mu \mid v) = \int \log(d\mu/dv)\, d\mu$.

The main result of this section is:

Theorem 4.1.

$$\limsup_N \frac{1}{N^d} H(\mu_t \mid v_t) = 0$$

The proof proceeds by means of several lemmas. The first of these is due to Yau [Y], and gives an upper bound for the time derivative of the relative entropy of μ_t to v_t:

Lemma 4.2.

$$\frac{d}{dt} \frac{1}{N^d} H(\mu_t \mid v_t) \leqslant \frac{1}{N^{d-1}} \int \frac{\mathscr{L}^*\psi}{\psi}\, d\mu_t - \frac{1}{N^d} \int \frac{\partial_t \psi}{\psi}\, d\mu_t$$

Proof. By definition, $H(\mu_t \mid v_t) = \int \log(d\mu_t/dv_t)\, d\mu_t = \int \log(f/\psi) f\, dv^c$; it follows that

$$\begin{aligned}
\partial_t H(\mu_t \mid v_t) &= \partial_t \int \log\left(\frac{f}{\varphi}\right) f\, dv^c \\
&= \int \left[\partial_t \log\left(\frac{f}{\psi}\right)\right] f\, dv^c + \int \log\left(\frac{f}{\psi}\right) \partial_t f\, dv^c \\
&= \int \partial_t \left(\frac{f}{\psi}\right) \psi\, dv^c + \int \log\left(\frac{f}{\psi}\right) (N\mathscr{L}^* f)\, dv^c \\
&= \int \partial_t f\, dv^c - \int \frac{\partial_t \psi}{\psi}\, d\mu_t + N \int \mathscr{L}\left[\log\left(\frac{f}{\psi}\right)\right] d\mu_t \\
&= 0 - \int \frac{\partial_t \psi}{\psi}\, d\mu_t + N \int \mathscr{L}\left[\log\left(\frac{f}{\psi}\right)\right] d\mu_t
\end{aligned}$$

Now,

$$\int \mathscr{L}\left[\log \frac{f}{\varphi}\right] d\mu_t$$

$$= \int \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u), \eta(v)) \left[\log \frac{f(\eta^{u,v})}{\psi(\eta^{u,v})} - \log \frac{f(\eta)}{f(\psi)}\right] d\mu_t$$

$$= \int \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u), \eta(v)) \log \frac{f(\eta^{u,v})\, \psi(\eta)}{f(\eta)\, \psi(\eta^{u,v})} f(\eta)\, dv^c$$

$$\leqslant \int \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u), \eta(v)) \left[\frac{f(\eta^{u,v})\, \psi(\eta)}{f(\eta)\, \psi(\eta^{u,v})} - 1\right] f(\eta)\, dv^c$$

$$= \int \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u), \eta(v)) \left[\frac{f(\eta^{u,v})}{\psi(\eta^{u,v})} - \frac{f(\eta)}{\psi(\eta)}\right] \psi(\eta)\, dv^c(\eta)$$

$$= \int \mathscr{L}\left(\frac{f}{\psi}\right)(\eta)\, \psi(\eta)\, dv^c$$

$$= \int \frac{f(\eta)}{\psi(\eta)} \mathscr{L}^* \psi(\eta)\, dv^c$$

$$= \int \frac{\mathscr{L}^* \psi(\eta)}{\psi(\eta)} d\mu_t$$

The result follows. ∎

We next calculate the adjoint operator $\mathscr{L}^*$.

Lemma 4.3.

$$\mathscr{L}^* f(\eta) = \sum_{u,v} p(v-u)\, \lambda\left(\frac{u}{N}\right) b(\eta(v), \eta(u))(f(\eta^{v,u}) - f(\eta))$$

$$-\frac{\gamma}{N} \cdot f(\eta) \sum_{u} \nabla_x \lambda\left(\frac{u}{N}\right) b(\eta(u), 0) + N^{d-1} f(\eta)\, R_N(\eta)$$

where $\lim_N R_N = 0$ uniformly in η.

Proof.

$$\int g(\eta)\,\mathscr{L}^*f(\eta)\,d\nu^c(\eta)$$

$$= \int f(\eta)\,\mathscr{L}g(\eta)\,d\nu^c(\eta)$$

$$= \int f(\eta) \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u),\eta(v))(g(\eta^{u,v})-g(\eta))\,d\nu^c(\eta)$$

$$= \int \Bigg[\sum_{u,v} f(\eta^{v,u})\, g(\eta)\,\lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u)+1,\eta(v)-1)\,\mathbb{1}(\eta(v)\neq 0)$$
$$\times \frac{d\nu^c(\eta^{v,u})}{d\nu^c(\eta)} - f(\eta)\,g(\eta)\,\lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(u),\eta(v))\Bigg]\,d\nu^c(\eta)$$

$$= \int g(\eta) \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u) \Bigg[f(\eta^{v,u})\, b(\eta(u)+1,\eta(v)-1)\,\mathbb{1}(\eta(v)\neq 0)$$
$$\cdot \frac{\Theta^c(\eta(u)+1)\,\Theta^c(\eta(v)-1)}{\Theta^c(\eta(u))\,\Theta^c(\eta(v))} - f(\eta)\, b(\eta(u),\eta(v))\Bigg]\,d\nu^c(\eta)$$

$$= \int g(\eta) \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u) \Bigg[f(\eta^{v,u})\, b(\eta(u)+1,\eta(v)-1)\,\frac{G(\eta(v))}{G(\eta(u)+1}$$
$$- f(\eta)\, b(\eta(u),\eta(v))\Bigg]\,d\nu^c(\eta)$$

From this and the detailed balance condition we deduce

$$\mathscr{L}^*f(\eta) = \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)[f(\eta^{v,u})\, b(\eta(v),\eta(u)) - f(\eta)\, b(\eta(u),\eta(v))]$$

$$= \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)(f(\eta^{v,u})-f(\eta))\, b(\eta(v),\eta(u))$$
$$- f(\eta) \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)[b(\eta(u),\eta(v)) - b(\eta(v),\eta(u))]$$

$$= \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)(f(\eta^{v,u})-f(\eta))\, b(\eta(v),\eta(u))$$
$$- f(\eta) \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)[b(\eta(u),0) - b(\eta(v),0)]$$

We then use the gradient condition and the assumption $\sum_v p(v) = 1$ to obtain

$$\begin{aligned}
\mathscr{L}^* f(\eta) &= \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)(f(\eta^{v,u}) - f(\eta))\, b(\eta(v), \eta(u)) \\
&\quad + f(\eta) \sum_u b(\eta(u), 0) \left[\sum_v p(u-v) \left(\lambda\left(\frac{v}{N}\right) - \lambda\left(\frac{u}{N}\right)\right)\right] \\
&= \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)(f(\eta^{v,u}) - f(\eta))\, b(\eta(v), \eta(u)) \\
&\quad + f(\eta) \sum_u b(\eta(u), 0) \left[\sum_v p(u-v) \frac{v-u}{N} \cdot \nabla_x \lambda\left(\frac{u}{N}\right) + \frac{1}{N} r_N(u)\right)\right]
\end{aligned}$$

for some r_N with $\lim_N r_N = 0$. This evidently completes the proof. ▮

Our next two lemmas deal with the two halves of the upper bound obtained in Lemma 4.2:

Lemma 4.4.

$$\begin{aligned}
\frac{1}{N^d} \int \frac{\partial_t \psi}{\psi}\, d\mu_t
&= -\frac{\gamma}{N^d} \cdot \int \sum_u \left\{ \lambda\left(\frac{u}{N}\right) \nabla_x m\left(t, \frac{u}{N}\right) h'\left(m\left(t, \frac{u}{N}\right)\right) \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} \right. \\
&\quad \times \left[M_{T_\ell(u)} \eta - m\left(t, \frac{u}{N}\right)\right] \\
&\quad \left. + \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}'\left(m\left(t, \frac{u}{N}\right)\right)\left[M_{T_\ell(u)} \eta - m\left(t, \frac{u}{N}\right)\right]\right\} d\mu_t + R_{N,\ell}
\end{aligned}$$

with $\lim_N R_{N,\ell} = 0$.

Proof.

$$\psi_t(\eta) = \Pi_u \frac{\Theta^{m(t, u/N)} \eta(u)}{\Theta^c \eta(u)}$$

$$\begin{aligned}
\frac{\partial_t \psi(\eta)}{\psi(\eta)} &= \sum_n \frac{\partial_t \Theta^{m(t, u/N)}(\eta(u))}{\Theta^{m(t, u/N)}(\eta(u))} \\
&= \sum_u \frac{\partial_t [(\alpha(m(t, u/N)))^{\eta(u)}\, \beta(m(t, u/N))/\tilde{G}(\eta(u))]}{(\alpha(m(t, u/N)))^{\eta(u)}\, \beta(m(t, u/N))/\tilde{G}(\eta(u))}
\end{aligned}$$

(using the fact that $\Theta^m(n) = [\alpha(m))^n \beta(m)]/\tilde{G}(n))$

$$= \sum_u \left[\eta(u) \frac{\partial_t \alpha(m(t, u/N))}{\alpha(m(t, u/N))} + \frac{\partial_t \beta(m(t, u/N))}{\beta(m(t, u/N))} \right]$$

$$= \sum_u \partial_t m\left(t, \frac{u}{N}\right) \left[\eta(u) \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} + \frac{\beta'(m(t, u/N))}{\beta(m(t, u/N))} \right]$$

$$= \sum_u \left[\partial_t m\left(t, \frac{u}{N}\right) \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} \right] \left[\eta(u) - m\left(t, \frac{u}{N}\right) \right]$$

(by Lemma 2.1)

$$= -\gamma \cdot \sum_u \nabla_x \left\{ \lambda\left(\frac{u}{N}\right) h\left(m\left(t, \frac{u}{N}\right)\right) \right\} \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} \left[\eta(u) - m\left(t, \frac{u}{N}\right) \right]$$

$$= -\gamma \cdot \sum_u \left[\nabla_x \lambda\left(\frac{u}{N}\right) h\left(m\left(t, \frac{u}{N}\right)\right) + \lambda\left(\frac{u}{N}\right) h'\left(m\left(t, \frac{u}{N}\right)\right) \nabla_x m\left(t, \frac{u}{N}\right) \right]$$
$$\times \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} \left[\eta(u) - m\left(t, \frac{u}{N}\right) \right]$$

$$= -\gamma \cdot \sum_u \left[\nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}'\left(m\left(t, \frac{u}{N}\right)\right) + \lambda\left(\frac{u}{N}\right) h'\left(m\left(t, \frac{u}{N}\right)\right) \nabla_x m\left(t, \frac{u}{N}\right) \right.$$
$$\left. \times \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} \right] \left[\eta(u) - m\left(t, \frac{u}{N}\right) \right]$$

$$= -\gamma \cdot \sum_u \left[\nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}'\left(m\left(t, \frac{u}{N}\right)\right) + \lambda\left(\frac{u}{N}\right) h'\left(m\left(t, \frac{u}{N}\right)\right) \nabla_x m\left(t, \frac{u}{N}\right) \right.$$
$$\left. \times \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))} \right] \left[M_{T_\ell(u)} \eta - m\left(t, \frac{u}{N}\right) + o(1) \right]$$

where for the last two equalities we used Lemma 2.2, the positivity of $\alpha(m)$, and the fact that the factor by which $\eta(u)$ is multiplied is continuous in u/N. The result follows.

Lemma 4.5.

$$\frac{1}{N^{d-1}} \int \frac{\mathscr{L}^* \psi}{\psi} d\mu_t = -\frac{\gamma}{N^d} \int \sum_u \left[\lambda\left(\frac{u}{N}\right) h(M_{T_\ell(u)} \eta) \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) \right.$$
$$\left. + \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}(M_{T_\ell(u)}) \right] du_t + R'_{N,L}$$

where $\lim_N R'_{N,\ell} = 0$.

Proof. We have

$$\frac{\psi(\eta^{v,u})}{\psi(\eta)} - 1 = \frac{\Theta^{m(t,\, v/N)}(\eta(v)-1)\ \Theta^{m(t,\, u/N)}(\eta(u)+1)\ \Theta^{c}(\eta(v))\ \Theta^{c}(\eta(u))}{\Theta^{m(t,\, v/N)}(\eta(v))\ \Theta^{m(t,\, u/N)}(\eta(u))\ \Theta^{c}(\eta(v)-1)\ \Theta^{c}(\eta(u)+1)} - 1$$

$$= \frac{1}{\alpha(m(t, v/N))} \cdot \frac{\alpha(m(t, u/N))}{1} \cdot \frac{\alpha(c)}{1} \cdot \frac{1}{\alpha(c)} - 1$$

(by the definition of $\Theta^{m(n))}$

$$= \frac{\alpha(m(t, u/N))}{\alpha(m(t, v/N))} - 1$$

$$= \frac{u-v}{N} \cdot \frac{\nabla_x \alpha(m(t, u/N))}{\alpha(m(t, u/N))} + O\left(\frac{1}{N^2}\right)$$

$$\frac{1}{N^{d-1}} \int \frac{\mathscr{L}^* \psi}{\psi}\, d\mu_t$$

$$= \frac{1}{N^{d-1}} \int \left\{ \sum_{u,\, v} \lambda\left(\frac{u}{N}\right) p(v-u)\, b(\eta(v), \eta(u)) \left(\frac{\psi(\eta^{v,u})}{\psi(\eta)} - 1\right) \right.$$

$$\left. - \frac{\gamma}{N} \cdot \sum_{u} \nabla_x \lambda\left(\frac{u}{N}\right) b(\eta(u), 0) \right\} d\mu_t + o(1)$$

$$= \frac{1}{N^{d-1}} \int \left\{ \sum_{u,v} \lambda\left(\frac{u}{N}\right) p(v-u)\, \frac{u-v}{N} \cdot b(\eta(v), \eta(u))\, \frac{\nabla_x \alpha(m(t, u/N))}{\alpha(m(t, u/m))} \right.$$

$$\left. + \sum_{u,v} \nabla_x \lambda\left(\frac{u}{N}\right) p(v-u) \cdot \frac{u-v}{N}\, b(\eta(u), 0) \right\} du_t + o(1)$$

(according to the calculation of $\psi(\eta^{v,u})/\psi(\eta)$ above, along with the fact that $\gamma = \sum_z z p(z)$)

$$= \frac{1}{N^{d-1}} \int \left\{ -\sum_{u,z} p(z)\, \frac{z}{N} \cdot \left[\lambda\left(\frac{u}{N}\right) M_{T_\ell(u)}\, b(\eta(\cdot + z), (\cdot))\, \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) \right.\right.$$

$$\left.\left. + \nabla_x \lambda\left(\frac{u}{N}\right) M_{T_\ell(u)}\, b(\eta(\cdot), 0) \right] \right\} d\mu_t + o(1)$$

$$= \frac{1}{N^{d-1}} \int - \sum_{u,z} p(z)\, \frac{z}{N} \cdot \left[\lambda\left(\frac{u}{N}\right) h(M_{T_\ell(u)} \eta)\, \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) \right.$$

$$\left. + \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}(M_{T_\ell(u)} \eta) \right] d\mu_t + o(1)$$

(by the one-block estimate)

$$= -\frac{\gamma}{N^d} \cdot \int \sum_u \left[\lambda\left(\frac{u}{N}\right) h(M_{T_\ell(u)}\eta)\, \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) + \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}(M_{T_\ell(u)}\eta) \right] d\mu_t + o(1)$$

The result follows.

This prepares us for

Proof of Theorem 4.1. By Lemmas 4.2, 4.4, and 4.5,

$$\frac{d}{dt} \frac{1}{N^d} H(\mu_t | \nu_t)$$

$$\leqslant \frac{1}{N^{d-1}} \int \frac{\mathscr{L}^* \psi}{\psi} d\mu_t - \frac{1}{N^d} \int \frac{\partial^t \psi}{\psi} d\mu_t$$

$$= -\int \frac{1}{N^d} \sum_u \left\{ \lambda\left(\frac{u}{N}\right) h(M_{T_\ell(u)}\eta) \gamma \cdot \nabla_x \log \left(m\left(t, \frac{u}{N}\right)\right) + \gamma \cdot \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}(M_{T_\ell(u)}\eta) - \lambda\left(\frac{u}{N}\right) \gamma \cdot \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) h'\left(m\left(t, \frac{u}{N}\right)\right)\left[M_{T_\ell(u)}\eta - m\left(t, \frac{u}{N}\right)\right] - \gamma \cdot \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}'\left(m\left(t, \frac{u}{N}\right)\right)\left[M_{T_\ell(u)}\eta - m\left(t, \frac{u}{N}\right)\right] \right\} du_t + o(1)$$

$$= -\int \frac{1}{N^d} \sum_u \left\{ \lambda\left(\frac{u}{N}\right) \gamma \cdot \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) + \left[h(M_{T_\ell(u)}\eta) - h'\left(m\left(t, \frac{u}{N}\right)\right)(M_{T_\ell(u)}\eta - m\left(t, \frac{u}{N}\right)) \right] + \gamma \cdot \nabla_x \lambda\left(\frac{u}{N}\right) \times \left[\hat{h}(M_{T_\ell(u)}\eta) - \hat{h}'\left(m\left(t, \frac{u}{N}\right)\right)\left(M_{T_\ell(u)}\eta - m\left(t, \frac{u}{N}\right)\right) \right] \right\} d\mu_t + o(1)$$

$$=: -\int \frac{1}{N^d} \sum_u X(u)\, d\mu_t + o(1)$$

We now split the integral into two cases on the value of $M_{T_\ell(u)}\eta$. We do this in order to use the Taylor approximation $|h(M)-h'(m)(M-m)-h(m)| \leqslant C(M-m)^2$ for some C, since $h''(M)$ is bounded for M less than any constant. As a result,

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{N^d} H(\mu_t \mid \nu_t) \leqslant & \frac{-1}{N^d} \int \sum_u \mathbb{1}(M_{T_\ell(u)}\eta \leqslant \sup m+1)\, X(u)\, d\mu_t \\
& -\frac{1}{N^d} \int \sum_u \mathbb{1}(M_{T_\ell(u)}\eta > \sup m+1)\, X(u)\, d\mu_t + o(1) \\
=: & \ \Omega_1 + \Omega_2 + o(1)
\end{aligned} \tag{4.1}
$$

Using the Taylor approximation for h and $\hat{h}$,

$$
\begin{aligned}
\Omega_1 \leqslant \frac{1}{N^d} \int \sum_u \Bigg\{ & -\mathbb{1}(M_{T_\ell(u)}\eta \leqslant \sup m+1) \\
& \times \left[\lambda\left(\frac{u}{N}\right) \gamma\cdot\nabla_x \log\alpha\left(m\left(t,\frac{u}{N}\right)\right) h\left(m\left(t,\frac{u}{N}\right)\right)\right. \\
& \left. + \gamma\cdot\nabla_x\lambda\left(\frac{u}{N}\right) \hat{h}\left(m\left(t,\frac{u}{N}\right)\right)\right] + C_1\left(M_{T_\ell(u)}\eta - m\left(t,\frac{u}{N}\right)\right)^2 \Bigg\}\, d\mu_t
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2 \leqslant & \frac{1}{N^d} \int \sum_u \mathbb{1}(M_{T_\ell(u)}\eta > \sup m+1) \\
& \times \left\{ \gamma\cdot\nabla_x \log\alpha\left(m\left(t,\frac{u}{N}\right)\right) \lambda\left(\frac{u}{N}\right) h'\left(m\left(t,\frac{u}{N}\right)\right)\right. \\
& \left. + \gamma\cdot\nabla_x\lambda\left(\frac{u}{N}\right) \hat{h}'\left(m\left(t,\frac{u}{N}\right)\right)\right\} \left(M_{T_\ell(u)}\eta - m\left(t,\frac{u}{N}\right)\right) du_t \\
& -\frac{1}{N^d} \int \sum_u \mathbb{1}(M_{T_\ell(u)}\eta > \sup m+1) \\
& \times \left\{ \gamma\cdot\nabla_x \log\alpha\left(m\left(t,\frac{u}{N}\right)\right) \lambda\left(\frac{u}{N}\right) h(M_{T_\ell(u)}\eta)\right. \\
& \left. + \gamma\cdot\nabla_x\lambda\left(\frac{u}{N}\right) \hat{h}(M_{T_\ell(u)}\eta)\right\} d\mu_t
\end{aligned}
$$

for some C_1. Hence the left-hand side of (4.1) is less than

$$
\begin{aligned}
& -\frac{1}{N^d} \int \sum_u \mathbb{1}(M_{T_\ell(u)}\eta \leqslant \sup m+1) \left\{\lambda\left(\frac{u}{N}\right) \gamma \cdot \nabla_x m\left(t, \frac{u}{N}\right) \frac{\alpha'(m(t, u/N))}{\alpha(m(t, u/N))}\right. \\
& \left.\times h\left(m\left(t, \frac{u}{N}\right)\right)+\gamma \cdot \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}\left(m\left(t, \frac{u}{N}\right)\right)\right\} \\
& +\frac{C_1}{N^d} \int \sum_u \left(M_{T_\ell(u)}\eta-m\left(t, \frac{u}{N}\right)\right)^2 d\mu_t \\
& +\frac{1}{N^d} \int \sum_u \gamma \cdot \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) \lambda\left(\frac{u}{N}\right) h'\left(m\left(t, \frac{u}{N}\right)\right) \\
& +\gamma \cdot \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}'\left(m\left(t, \frac{u}{N}\right)\right) \Big| \left(M_{T_\ell(u)}\eta-m\left(t, \frac{u}{N}\right)\right)^2 du_t \\
& +\frac{1}{N^d} \int \sum_u \Big| \gamma \cdot \nabla_x \log \alpha\left(m\left(t, \frac{u}{N}\right)\right) \lambda\left(\frac{u}{N}\right) h(M_{T_\ell(u)}\eta) \\
& +\gamma \cdot \nabla_x \lambda\left(\frac{u}{N}\right) \hat{h}] M_{T_\ell(u)}\eta) \Big| \left(M_{T_\ell(u)}\eta-m\left(t, \frac{u}{N}\right)\right)^2 d\mu_t+o(1)
\end{aligned}
$$

(bounding $\mathbb{1}(M_{T_\ell(u)}\eta - m(t, u/N))$ by $(M_{T_\ell(u)}\eta - m(t, u/N))^2$ in the second term and $\mathbb{1}(M_{T_\ell(u)}\eta > \sup m+1)$ by $(M_{T_\ell(u)}\eta - m(t, u/N))^2$ in the third)

$$
\begin{aligned}
\leqslant & -\frac{1}{N^d} \int \sum_u \gamma \cdot \nabla_x\{\lambda(\cdot)\, \hat{h}(m(t, \cdot))\}\left(\frac{u}{N}\right) d\mu_t \\
& +\frac{1}{N^d} \int \sum_u \mathbb{1}(M_{T_\ell}\eta > \sup m+1) \gamma \cdot \nabla_x\{\lambda(\cdot)\, \hat{h}(m(t, \cdot))\}\left(\frac{u}{N}\right) d\mu_t \\
& +\frac{C^2}{N^d} \int \sum_u \left(M_{T_\ell(u)}\eta-m\left(t, \frac{u}{N}\right)\right)^2 d\mu_t+o(1) \\
\leqslant & -\frac{1}{N^d} \int \sum_u \gamma \cdot \nabla_x\{\lambda(\cdot)\, \hat{h}(m(t, \cdot))\}\left(\frac{u}{N}\right) d\mu_t \\
& +\frac{C^3}{N^d} \int \sum_u \left(M_{T_\ell(u)}\eta-m\left(t, \frac{u}{N}\right)\right)^2 d\mu_t+o(1)
\end{aligned}
$$

(for some C_2 and C_3 applying Lemma 2.2 to the first term and using the fact that all the factors by which $(M-m)^2$ is multiplied in the last two terms of the pevious expression are bounded).

The first term inside the integral vanishes as $N\to\infty$, since it approximates an integral of the exact derivative $\gamma\cdot\nabla_x\{\lambda\cdot\hat{h}(m(t,\cdot))\}$. It follows that

$$\begin{aligned}
&\limsup_N \frac{1}{N^d}\frac{d}{dt}H(\mu_t\mid\nu_t)\\
&\quad\leqslant \limsup_{\ell}\limsup_N \frac{C_3}{N^d}\int\sum_u\left(M_{T_\ell(u)}\eta - m\left(t,\frac{u}{N}\right)\right)^2 d\mu_t\\
&\quad\leqslant \limsup_{\ell}\limsup_N \frac{C_4}{N^d}\int\phi\left(M_{T_\ell(u)}\eta, m\left(t,\frac{u}{N}\right)\right)d\mu_t \quad (\text{for some } C_4)
\end{aligned}$$

since the microscopic entropy function ϕ, defined in Section 2, is uniformly convex. By Lemma 4.6 below, this is bounded by $\limsup_\ell\limsup_N C_3/N$ $H(\mu_t\mid\nu_t)$; by Gronwall's Inequality, the result follows.

It therefore remains only to show

Lemma 4.6.

$$\begin{aligned}
&\limsup_{\ell}\limsup_N \int\frac{1}{N^d}\sum_u\phi\left(M_{T_\ell(u)}\eta, m\left(t,\frac{u}{N}\right)\right)d\mu_t\\
&\quad\leqslant \limsup_N \frac{2}{N}H(\mu_t\mid\nu_t)
\end{aligned}\tag{4.2}$$

Proof. By the entropy inequality,

$$\begin{aligned}
&\frac{1}{N^d}\int\sum_u\phi\left(M_{T_\ell(u)}\eta, m\left(t,\frac{u}{N}\right)\right)d\mu_t\\
&\quad\leqslant \frac{2}{N^d}H(\mu_t\mid\nu_t)+\frac{1}{N^d}\log\int\exp\left[\frac{1}{2}\sum_u\phi\left(M_{T_\ell(u)}\eta, m\left(t,\frac{u}{N}\right)\right)\right]d\nu_t
\end{aligned}$$

We must show that the latter term is bounded above by zero in the limit.

Let $\{u_j\}$ be chosen so that $\{T_\ell(u_j)\}$ covers $\mathbb{Z}_N^d$ disjointly; let

$$X_z=\frac{1}{2}\sum_j\phi\left(M_{T_\ell(u_j+z)}\eta, m\left(t,\frac{u_j+z}{N}\right)\right)$$

for $z \in T_\ell$. We then have

$$\frac{1}{N^d}\log\int\exp\left[\frac{1}{2}\sum_u \phi\left(M_{T_\ell(u)}\eta, m\left(t,\frac{u}{N}\right)\right)\right]d\nu_t$$

$$=\frac{1}{N^d}\log\int\exp\left(\sum_{z\in T_\ell} X_2\right)d\nu_t$$

$$\leqslant\frac{1}{N^d}\frac{1}{|T_\ell|}\sum_z\log\int\exp(|T_\ell|\,X_z)\,d\nu_t=:\frac{1}{|T_\ell|}\sum_z\Omega_z$$

by the convexity of $f\mapsto\log\int\exp f\,d\nu_t$. Moreover, for each z

$$\Omega_z=\frac{1}{N^d}\log\int\exp\left[|T_\ell|\sum_j\phi\left(M_{T_\ell(u_j+z)}\eta, m\left(t,\frac{u_j+z}{N}\right)\right)\right]d\nu_t$$

$$=\frac{1}{N^d}\log\int\Pi_j\exp\left[|T_\ell|\,\phi\left(M_{T_\ell(u_j+z)}\eta, m\left(t,\frac{u_j+z}{N}\right)\right)\right]d\nu_t$$

$$=\frac{1}{N^d}\sum_j\log\int\exp\left[|T_\ell|\,\phi\left(M_{T_\ell(u_j+z)}\eta, m\left(t,\frac{u_j+z}{N}\right)\right)\right]d\nu_t \tag{4.3}$$

since ν_t is a product measure. Set $m_j=m(t,(u_j+z)/N)$ and let $\nu_{t,j}$ and $\nu_j^{m_j}$ denote the restriction of measures ν_t and ν^{m_j} to the block $T_\ell(u_j+z)$. From (4.3) and Schwartz inequality we obtain

$$\Omega_z\leqslant\frac{1}{2N^d}\sum_j\log\int\exp\left[tT_\ell|\,\phi\left(M_{T_\ell(u_j+z)}\eta, m\left(t,\frac{u^j+z}{N}\right)\right)\right]d\nu^{m_j}$$

$$+\frac{1}{2N^d}\sum_j\log\int\left(\frac{\nu_{t,j}}{\nu^{m_j}}\right)^2d\nu^{m_j} \tag{4.4}$$

Since m is smooth, it is not hard to show that the second term on the right-hand side of (4.4) is of order $O(\ell^d/N^d)$. Hence

$$\Omega_z\leqslant\frac{1}{2N^d}\sum_j\log\left\{\sum_{k=0}^{|T_\ell|}\exp\left[|T_\ell|\,\phi\left(\frac{k}{|T_\ell|}, m\left(t,\frac{u_j+z}{N}\right)\right)\right]\right.$$
$$\left.\times\,\nu^{m_j}\left(M_{T_\ell(u_j+z)}\eta=\frac{k}{|T_\ell|}\right)\right\}+O\left(\frac{\ell^d}{N^d}\right)$$

(since ν_t is a product of measures $\Theta^{m(t,u/N)}$, each of which is within order ℓ/N of $\Theta^{m(t,u_j/N)}$. The measure $\nu^{m(t,u_j/N)}$ is a product of this latter measure).

We must now calculate $v^m(M\eta = k/|T_\ell|)$. But by Lemma 2.3, we have

$$v^m\left(M\eta = \frac{k}{|T_\ell|}\right) \leqslant \exp\left[-|T_\ell|\,\phi\left(\frac{k}{|T_\ell|}, m\right)\right]$$

Therefore, the left-hand side of (4.2) is bounded above by

$$\limsup_{\ell}\limsup_{N}\sup_{u}\frac{1}{N^d}\sum_j \log\left\{\sum_{k=0}^{|T_\ell|}\exp\left[|T_\ell|\,\phi\left(\frac{k}{|T_\ell|}, m\left(t,\frac{u}{N}\right)\right)\right]\right.$$
$$\times\exp\left[-|T_\ell|\,\phi\left(\frac{k}{|T_\ell|} m\left(t,\frac{u}{N}\right)\right)\right]$$
$$\left.=\limsup_{\ell}\limsup_{N}\frac{1}{N^d}\frac{N^d}{|T_\ell|}\log(|T_\ell|+1)\right\}=0$$

This concludes the proof of Lemma 4.6.

From this relative entropy estimate, we may now reach our desired conclusion about the particle density profiles of the process:

Theorem 4.7 (Periodic Version). Suppose (1.3) has a smooth solution m on $[0, T]\times\mathbb{T}^d$ for which $\alpha(m)>0$; then a periodic process with initial density profile $m(0, x)$ will have density profile $m(t, x)$ at later times $t\in[0, T]$.

Proof. By Lemma 4.6 and Theorem 4.1,

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\int\frac{1}{N^d}\sum_u \phi\left(M_{T_\ell(u)}\eta_t, m\left(t,\frac{u}{N}\right)\right)d\mu^t=0$$

according to the uniform convexity of ϕ,

$$\left(M_{T_\ell(u)}\eta^t - m\left(t,\frac{u}{N}\right)\right)^2 \leqslant C\phi\left(M_{T_\ell(u)}\eta_t, m\left(t,\frac{u}{N}\right)\right)$$

for some C, and so

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\int\frac{1}{N^d}\sum_u\left(M_{T_\ell(u)}\eta_t - m\left(t,\frac{u}{N}\right)\right)^2 d\mu_t=0$$

From this it follows that the process η_t with law μ_t^N does indeed have density profile $m(t, u/N)$. ∎

This result further extends to

Theorem 4.7 (Nonperiodic Version). Suppose (1.3) has a smooth solution m on $[0, T] \times \mathbb{R}^d$ for which $\alpha(m) > 0$. Then a PdM with initial density profile $m(0, x)$ will have density profile $m(t, x)$ at later times $t \in [0, T]$.

Proof. This is because, since particles are propagated with finite speed with probability 1, Lemma 5.7 of [R] tells us that for any macroscopic box of length (for example) $\frac{1}{2}$ in each spatial dimension, we can couple a periodic process to a non-periodic one in such a way that no discrepancies appear in that box for some fixed positive time independent of the location of the box. Thus the particle densities match $m(t, x)$ in any box of that size for some positive time; piecing together boxes, we draw the same conclusion for all of $\mathbb{R}^d$; and by repeating the argument, we find that the result holds for the whole time interval $[T_1, T_2]$.

5. MONOTONICITY AND MICROSCOPIC ENTROPY INEQUALITY

In this section, we use the *monotonicity* property of the PdM to derive a certain microscopic version of Kružkov's entropy inequality for the process η_t.

The idea behind monotonicity is this: we can couple together two PdM's having the same generator in such a way that the number of discrepancies between the processes (that is, the sum $\sum_u |\eta(u) - \zeta(u)|$ for the processes (η, ζ)) cannot increase over time as the generator acts on the processes. The generator $\tilde{\mathscr{L}}^N$ is defined for a pair of configurations (η, ζ) as follows:

$$
\begin{aligned}
&\tilde{\mathscr{L}}^N f(\eta, \zeta) \\
&\quad = \sum_{u,v} p(v-u)\, \lambda\left(\frac{u}{N}\right) \{[b(\eta(u), \eta(v)) \wedge b(\zeta(u), \zeta(v))](f(\eta^{u,v}, \zeta^{u,v}) - f(\eta, \zeta)) \\
&\qquad + [b(\eta(u), \eta(v)) - b(\eta(u), \eta(v)) \wedge b(\zeta(u), \zeta(v))](f(\eta^{u,v}, \zeta) - f(\eta, \zeta)) \\
&\qquad + [b(\zeta(u), \zeta(v)) - b(\eta(u), \eta(v)) \wedge b(\zeta(u), \zeta(v))](f(\eta, \zeta^{u,v}) - f(\eta, \zeta))\}
\end{aligned}
$$

When λ is identically one, the generator is denoted by $\tilde{\mathscr{L}}^{(1)}$.

Let $\tilde{\mathscr{I}}$ be the space of invariant measures under the generator $\tilde{\mathscr{L}}^{(1)}$, and let $\tilde{\mathscr{S}}$ be the space of measures that the shift-invariant on the space of pairs of configurations. Note that the coupled generator restricted to either process in isolation reduces to the ordinary PdM generator.

We note that the same argument that proved our original Lemma 3.1 also proves

Lemma 3.1 (Coupled Version). Let $\tilde{S}_t^N$ be the semigroup corresponding to the generator $N\tilde{\mathscr{L}}^N$. Let $\tilde{\mu}_t^N = \tilde{S}_t^N * \tilde{\mu}_0^N$, where $\tilde{\mu}_0^N$ is the initial measure on the space of pairs of configurations. Write

$$\hat{\mu}^N = \frac{1}{N^{d+1}} \sum_{|u| \leqslant k} \int_0^T \tau_{-u} \tilde{\mu}_t^N \, dt$$

let μ^* be a limit point of $\{\hat{\mu}^N\}$. Then, for any local f,

$$\int \tilde{\mathscr{L}}^{(1)} f \, d\mu^* = 0$$

This lemma will be used in proving the main result of this section, which is

Theorem 5.1. Let m be a smooth solution of (1.3) on $[T_1, T_2]$ with $\alpha(m) > 0$; let η_t be generated by the ordinary PdM generator $N\mathscr{L}$. Then

$$\lim_{\ell \to \infty} \liminf_{N \to \infty} \mu^N \left\{ \int_{T_1}^{T_2} \frac{1}{N^d} \sum_u \partial_s J\left(s, \frac{u}{N}\right) \left| M_{T_l(u)} \eta_s - m\left(s, \frac{u}{N}\right) \right| ds \right.$$
$$\left. + \int_{T_2}^{T_2} \frac{1}{N^d} \sum_u \gamma \cdot \nabla_x J\left(s, \frac{u}{N}\right) q\left(M_{T_l(u)} \eta_s; m\left(s, \frac{u}{N}\right)\right) ds \geqslant -\varepsilon \right\} = 1$$

Proof. The proof is sketched because it is very similar to the proof of Theorem 3.1 of [R]. (We assume without loss of generality that $T_1 = 0$, $T_2 = T$.) By Theorem 4.7, the function m can be obtained as the density profile of a Markov process ζ_t generated by $N\mathscr{L}$, initially distributed as ν_0^N where ν_0^N is as in the previous section. Let (η_t, ζ_t) be a coupled process with initial distribution $\mu^N \times \nu_0^N$, generated by $N\mathscr{L}$. Denote the law of this process by $\tilde{P}^N$. Choose a test function J; choose a constant $T > \sup\{\mathrm{supp}_t J\}$. Following [R], one can readily arrive at

$$\lim_N \tilde{P}^N \left\{ \int_0^T \frac{1}{N^d} \sum_u \partial_t J\left(t, \frac{u}{N}\right) |\eta_t(u) - \zeta^t(u)| \right.$$
$$\left. + \nabla_x J\left(t, \frac{u}{N}\right) \lambda\left(\frac{u}{N}\right) \tau_u H(\eta_t, \zeta_t) \Big] \, dt \geqslant -\varepsilon \right\} = 1$$

where $H(\eta, \zeta) = \sum_z z p(z)(b(\zeta(0), \zeta(z)) - b(\eta(0), \eta(z)))(F_{0,z}(\eta, \zeta) - F_{0,z}(\zeta, \eta))$ and $F_{u,v}(\eta, \zeta) = 1_{\eta(u) \geqslant \zeta(u), \eta(v) \geqslant \zeta(v)}$.

Again, as in [R], we can use Lemma 5.1 to replace $|\eta_t(u)-\zeta_t(u)|$ by $|M_{T_\ell(u)}\eta_t - M_{T_\ell(u)}\zeta_t|$ and $\tau_u H(\eta_t, \zeta_t)$ by $q(M_{T_\ell(u)}\eta_t;\, M_{T_\ell(u)}\zeta_t)$ in the above statement. For the former we need somthing like

$$\lim_{\bar{\rho}\to\infty}\ \liminf_{N\to\infty} E^N\left[\frac{1}{N^d}\sum_u \mathbb{1}(\eta_t(u)\geqslant\bar{\rho}),\, \zeta_t(u)\geqslant\bar{\rho}\right]=0 \tag{5.1}$$

to allow us to replace $|\eta(u)-\zeta(u)|$ with the uniformly bounded function

$$|\eta(u)-\zeta(u)|\ \mathbb{1}(\eta(u)<\bar{\rho},\, \zeta(u)<\bar{\rho})$$

The statement (5.1) is a straightforward consequence of Chebyshev's inequality and the integrability of ρ_0. Finally we apply Theorem 4.7 to replace M_{T_ℓ} with m and this completes the proof. ▮

6. PDE LEMMAS AND PROOF OF THEOREM 1.2

In this section, we will supply the missing link in our proof of Theorem 1.3 by means of the concept of *measure-valued solutions*, as presented by DiPerna [D]. For each (t, x) in $[0, \infty)\times\mathbb{R}^d$, let $\pi_{t,x}$ be a measure on $[0, \infty)$. We then define π to be a *measure-valued solution* of (1.3) if the following holds:

$$\int [\partial_t J(t, x)k + \nabla_x J(t, x)\, \lambda(x)\, h(k)]\ \pi_{t,x}(dk)\, dx\, dt = 0$$

for all test functions J.

We approach the definition of a measure π corresponding to the process η_t as follows: we first define the *Young measure* $\pi^{N,\ell}(t, dx; dk)$ by

$$\int G(x, k)\, \pi^{N,\ell}(t, dx; dk) = \frac{1}{N^d}\sum_u G\left(\frac{u}{N}, M_{T_\ell(u)}\eta_t\right)$$

We then use the correspondence between η_t and $\pi^{N,\ell}(t, \cdot; \cdot)$ to define inductively the associated probability measures $R^{N,\ell}(t, dx; dk)$.

The sequence $R^{N,\ell}$ is tight, passinf to the limit first in N and then in ℓ. Let R be any limit point. Then, R-almost surely, there exists a family of measures $\pi_{t,x}(dk)$ with $\pi_{t,x}(dk)\, dx = \pi(t, dx; dk)$. (See [R], Lemma 5.5.)

We can translate our previous results into the language of measure-valued solutions using the fact that $\pi_{t,x}$ is the limiting probability distribution of microscopic block particle densities near point x at time t. For example, Theorem 5.1 is equivalent to the following:

Theorem 5.1 (Restated). Let $m(t, x)$ be a smooth solution of (1.3) in $[T_1, T_2]$ with $\alpha(m)>0$. Then, for any test function J with support in (T_1, T_2),

$$\int \{\partial_t J(t, x)\,|k-m(t, x)| + \gamma\cdot\nabla_x J(t, x)\,\lambda(x)\,q(k; m(t, x))\}\,\pi_{t,x}(dk)\,dx\,dt \geqslant 0$$

with probability 1 according to the measure R.

To this end, we first prove a pair of PDE lemmas concerning measure-valued solutions.

Lemma 6.1. Suppose that

$$\int [\partial_t J(t, x)\,|k-m(t, x)| + \gamma\cdot\nabla_x J(t, x)\,\lambda(x)\,q(k; m(t, x))]\,\pi_{t,x}(dk)\,dx\,dt \geqslant 0$$

for all smooth solutions m of the PDE (1.3) on $[T_1, T_2]\times\mathbb{T}^d$ with $\alpha(m)>0$ and all test functions J supported on the interior of that domain. Then, for all constants c_0 and all test functions J of compact support,

$$\begin{aligned}\int [\partial_t J(t, x)\,|k-c_0| &+ \gamma\cdot\nabla_x J(t, x)\,\lambda(x)\,q(k; c_0)\\ &- J(t, x)\,\mathrm{sgn}(k-c_0)\gamma\cdot\nabla\lambda(x)\,h(c_0)]\,\pi_{t,x}(dk)\,dx\,dt \geqslant 0 \qquad (6.1)\end{aligned}$$

Proof. Note that it suffices to establish (6.1) for constants c_0 with $\alpha(c_0)>0$; the inequality (6.1) in the case $\alpha(c_0)$ is established by approximation. Choose a constant c_0 with $\alpha(c_0)>0$. For every positive s_0, let $m(t, x; s)$ be such that $\alpha(m)>0$ and for $s\in(s_0-\delta, s_0+\delta)$

$$\begin{cases} m(s, x; s) = c_0 & \text{for} \quad x\in\mathbb{T}^d \\ m(t, x; s) & \text{a smooth solution of (1.3) for} \quad t\in(s_0-\delta, s_0+\delta) \end{cases}$$

For sufficiently small δ such m exists and depends smoothly on s. (This can be done, since m is composed of solutions to characteristic ODE's.) Let H be a mollifier function on $\mathbb{R}$ (that is: H smooth, $H\geqslant 0$, $\int H=1$, $\operatorname{supp} H$ bounded); let $H^\varepsilon(z)=(1/\varepsilon)\,H(z/\varepsilon)$. Let $J^\varepsilon(t, x; s)=J((t+s)/2, x)\,H^\varepsilon((t-s)/2)$ where J is a smooth function with support in $(s_0-\delta, s_0+\delta)-\mathbb{T}^d$.

Since $m(\cdot, \cdot; s)$ is a smooth solution, our supposition leads to

$$\begin{aligned}\int [\partial_t J^\varepsilon(t, x; s)\,&|k-m(t, x; s)|\\ &+\gamma\cdot\nabla_x J^\varepsilon(t, x; s)\,\lambda(x)\,q(k; m(t, x; s))]\,\pi_{t,x}(dk)\,dx\,dt\,ds \geqslant 0 \qquad (6.2)\end{aligned}$$

Furthermore,

$$\Omega_\varepsilon := \int \partial_t J^\varepsilon(t, x; s)\, |k - m(t, x; s)|\, \pi_{t,x}(dk)\, dx\, dt\, ds$$

$$= \int \frac{1}{2} (\partial_t J)\left(\frac{t+s}{2}, x\right) H^\varepsilon\left(\frac{t-s}{2}\right) |k - m(t, x; s)|\, ds\, \pi_{t,x}(dk)\, dx\, dt$$

$$+ \int \frac{1}{2} J\left(\frac{t+s}{2}, x\right) (H^\varepsilon)'\left(\frac{t-s}{2}\right) |k - m(t, sx; s)|\, ds\, \pi_{t,x}(dk)\, dx\, dt$$

We integrate by parts for the ds-integral in the second term to get

$$\frac{1}{2} \int (\partial_t J)\left(\frac{t+s}{2}, x\right) H^\varepsilon\left(\frac{t-s}{2}\right) |k - m(t, x; s)|\, ds\, \pi_{t,x}(dk)\, dx\, dt$$

$$+ \frac{1}{2} \int (\partial_t J)\left(\frac{t+s}{2}, x\right) H^\varepsilon\left(\frac{t-s}{2}\right) |k - m(t, x; s)|\, ds\, \pi_{t,x}(dk)\, dt$$

$$- \int J\left(\frac{t+s}{2}, x\right) H^\varepsilon\left(\frac{t-s}{2}\right) \operatorname{sgn}(k - m(t, x; s))\, \partial_s m(t, x; s)\, ds\, \pi_{t,x}(dk)\, dx\, dt$$

We let ε goes to zero. As a result,

$$\lim_\varepsilon \Omega_\varepsilon = 2 \int (\partial_t J)(t, x)\, |k - c_0|\, \pi_{t,x}(dk)\, dx\, dt$$

$$- 2 \int J(t, x) \operatorname{sgn}(k - c_0)\, \partial_s m(t, x; t)\, \pi_{t,x}(dk)\, dx\, dt$$

On the other hand, since $m(s, x; s) = c_0$, we have $\nabla_x m(s, x; s) = 0$ and

$$\partial_s m(s, x; s) = -\partial_t m(s, x; s) = \gamma \cdot \nabla_x(\lambda(x)\, h(m(s, x; s)))$$

because $m(\cdot, \cdot; s)$ is a solution to (3.1). Thus the limit of the first term in (6.2) equals

$$2 \int (\partial_t J)(t, x)\, |k - c_0|\, \pi_{t,x}(dk)\, dx\, dt - 2 \int J(t, x) \operatorname{sgn}(k - c_0)\, \nabla_x \lambda(x)\, h(c^0) \tag{6.3}$$

It is not hard to see that the limit of the second term in (6.2) equals to

$$2\gamma \cdot \nabla_x J(t, x)\, \lambda(x)\, q(k; c_0)\, \pi_{t,x}(dk)\, dx\, dt$$

This and (6.3) combined with (6.2) complete the proof of lemma.

Having begun with an enropy inequality comparing $\pi_{t,x}(dk)$ to smooth solutions $m(t, x)$ and derived another one comparing it to constants c_0, we next use this result to compare $\pi_{t,x}(dk)$ with any distribution solution $\rho(t, x)$.

Lemma 6.2. Suppose that for all test functions J with support in $(0, T) \times \mathbb{T}^d$ and all constants c_0 the inequality (6.1) holds; then for all distribution solutions ρ of the PDE (1.3) on $[0, T] \times \mathbb{T}^d$;

$$\int [\partial_t J(t, x)\, |k - \rho(t, x)| + \gamma \cdot \nabla_x J(t, x)\, \lambda(x)\, q(k; \rho(t, x))]\, \pi_{t,x}(dk)\, dx\, dt \geqslant 0 \tag{6.4}$$

Proof. For this result, we will be using mollifiers even more extensively. Define $J^\varepsilon(t, x, s, y) = J((t+s)/2, \quad (x+y)/2)\, H^\varepsilon((t-s)/2) \quad H^\varepsilon_d((x-y)/2)$ where H^ε is as in the proof of Lemma 6.2 and $H^\varepsilon_d(z) = \prod_{i=1}^d H^\varepsilon(z_i)$. We certainly have

$$(\partial_t + \partial_s)\, J^\varepsilon(t, x; s) = (\partial_t J)\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^\varepsilon\left(\frac{t-s}{2}\right) H^\varepsilon_d\left(\frac{x-y}{2}\right)$$

$$\gamma \cdot (\nabla_x + \nabla_y)\, J^\varepsilon(t, x, s, y) = \gamma \cdot (\nabla_x J)\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^\varepsilon\left(\frac{t-s}{2}\right) H^\varepsilon_d\left(\frac{x-y}{2}\right) \tag{6.5}$$

Moreover,

$$\begin{aligned}
&\int [\partial_t J(t, x)\, |k - \rho(t, x)| + \gamma \cdot \nabla_x J(t, x)\, \lambda(x)\, q(k; \rho(t, x))]\, \pi_{t,x}(dk)\, dx\, dt \\
&= \lim_{\varepsilon \to 0} \frac{1}{2^{d+1}} \int \Bigg[\partial_t J\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^\varepsilon\left(\frac{t-s}{2}\right) H^\varepsilon_d\left(\frac{x-y}{2}\right) |k - \rho(t, x)| \\
&\quad + \gamma \cdot \nabla_x J\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^\varepsilon\left(\frac{t-s}{2}\right) H^\varepsilon_d\left(\frac{x-y}{2}\right) \lambda(x)\, q(k; \rho(t, x)) \Bigg] \\
&\quad \times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt \\
&= \lim_{\varepsilon \to 0} \frac{1}{2^{d+1}} \int \Bigg[\partial_t J\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^\varepsilon\left(\frac{t-s}{2}\right) H^\varepsilon_d\left(\frac{x-y}{2}\right) |k - \rho(s, y)| \\
&\quad + \gamma \cdot \nabla_x J\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^\varepsilon\left(\frac{t-s}{2}\right) H^\varepsilon_d\left(\frac{x-y}{2}\right) \lambda(x)\, q(k; \rho(s, y)) \Bigg] \\
&\quad \times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt
\end{aligned} \tag{6.6}$$

where the second equality follows from the fact that both $|k-z|$ and $q(k; z)$ are Lipshitz functions in z and that

$$\lim_{\varepsilon \to 0} \int \left| \partial_t J\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \right| H^\varepsilon\left(\frac{t-s}{2}\right) H_d^\varepsilon\left(\frac{x-y}{2}\right) |\rho(t, x) - \rho(s, y)|$$

$$\times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt = 0 \tag{6.7}$$

In fact (6.7) is a consequence of $\int d\pi_{x,t}(dk) = 1$, the fact that

$$\lim_{\alpha, \beta \to 0} \int_{|x| \leqslant M} \int_0^T |\rho(t+\beta, x+\alpha) - \rho(t, x)|\, dx\, dt = 0$$

and the bounded convergence theorem. From (6.5) we deduce that the right-hand side of (6.6) equals to

$$\frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int [(\partial_t + \partial_s)\, J^\varepsilon(t, x, s, y)\, |k - \rho(s, y)|$$

$$+ \gamma \cdot (\nabla_x + \nabla_y)\, J^\varepsilon(t, x, s, y)\, \lambda(x)\, q(k; \rho(s, y))]\, \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$= \frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int [\partial_t J^\varepsilon(t, x, s, y)\, |k - \rho(s, y)|$$

$$+ \gamma \cdot \nabla_x J^\varepsilon(t, x, s, y)\, \lambda(x)\, q(k; \rho(x, y))]\, \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$+ \frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int [\partial_s J^\varepsilon(t, x, s, y)\, |k - \rho(s, y)|$$

$$+ \gamma \cdot \nabla_y J^\varepsilon(t, x, s, y)\, \lambda(y)\, q(k; \rho(s, y))]\, \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$+ \frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int \gamma \cdot \nabla_y J^\varepsilon(t, x, s, y)(\lambda(x) - \lambda(y))\, q(k; \rho(s, y))$$

$$\times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

(rearranging the terms)

$$\geqslant \frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int J^\varepsilon(t, x, s, y)[\operatorname{sgn}(k - \rho(s, y))\, h(\rho(s, y))]\, \gamma \cdot \nabla \lambda(x)$$

$$\times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$+\frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int J^{\varepsilon}(t, x, s, y)[\operatorname{sgn}(\rho(s, y)-k)\, h(k)]\, \gamma \cdot \nabla \lambda(y)$$

$$\times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$+\frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int \gamma \cdot \nabla_y J^{\varepsilon}(t, x, s, y)(\lambda(x)-\lambda(y))\, q(k; \rho(s, y))$$

$$\times \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$=: \Omega_1+\Omega_2+\Omega_3$$

using the entropy inequality (6.1) on the first term and (1.4) on the second.

It is not hard to show

$$\Omega_1+\Omega_2=\int h(t, x) \operatorname{sgn}(k-\rho(t, x))(h(\rho(t, x))-h(k))\gamma \cdot \nabla \lambda(x)\, \pi_{t,x}(dk)\, dx\, dt \tag{6.8}$$

In fact for every Lebsegue point (t, x) of ρ, we have

$$\frac{1}{2^{d+1}} \lim_{\varepsilon \to 0} \int J^{\varepsilon}(t, x, s, y)[\operatorname{sgn}(k-\rho(s, y))\, h(\rho(s, y))]\, \gamma \cdot \nabla \lambda(x)\, dy\, ds$$

$$=\int J(t, x) \operatorname{sgn}(k-\rho(t, x))\, h(\rho(t, x))\gamma \cdot \nabla \lambda(x) \tag{6.9}$$

Using (6.9) and the bounded convergenge theorem, one can readily establish (6.8).

Moreover

$$\Omega_3=\lim_{\varepsilon \to 0} \frac{1}{2^{d+2}} \gamma \cdot \int (\nabla_x J)\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^{\varepsilon}\left(\frac{t-s}{2}\right) H_d^{\varepsilon}\left(\frac{x-y}{2}\right)$$

$$\times (\lambda(x)-\lambda(y))\, q(k; \rho(s, y))]\, \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$-\lim_{\varepsilon \to 0} \frac{1}{2^{d+2}} \gamma \cdot \int J\left(\frac{t+s}{2}, \frac{x+y}{2}\right) H^{\varepsilon}\left(\frac{t-s}{2}\right) (\nabla H_d^{\varepsilon}) \left(\frac{x-y}{2}\right)$$

$$\times (\lambda(x)-\lambda(y))\, q(k; \rho(s, y))]\, \pi_{t,x}(dk)\, dy\, ds\, dx\, dt$$

$$=: \Omega_{31}+\Omega_{32}$$

Since $\lambda(x)-\lambda(y)$ vanishes as x approaches y, it is not hard to show that $\Omega_{31}=0$. On the other hand,

$$\Omega_{32}=\lim_{\varepsilon\to 0}\frac{1}{2^{d+2}}\gamma\cdot\int J\left(\frac{t+s}{2},\frac{x+y}{2}\right)H^{\varepsilon}\left(\frac{t-s}{2}\right)(\nabla H^{\varepsilon}_{d})\left(\frac{x-y}{2}\right)$$

$$\times(\lambda(x)-\lambda(y))\,q(k;\rho(t,x))]\,\pi_{t,x}(dk)\,dy\,ds\,dx\,dt$$

$$+\lim_{\varepsilon\to 0}\frac{1}{2^{d+2}}\gamma\cdot\int J\left(\frac{t+s}{2},\frac{x+y}{2}\right)H^{\varepsilon}\left(\frac{t-s}{2}\right)(\nabla H^{\varepsilon}_{d})\left(\frac{x-y}{2}\right)$$

$$\times(\lambda(x)-\lambda(y))[q(k;\rho(s,y)-q(k;\rho(t,x))]\,\pi_{t,x}(dk)\,dy\,ds\,dx\,dt$$

$$=:\Omega_{321}+\Omega_{322}$$

Moreover, since q is Lipschitz and λ is continuously differentiable,

$$|\Omega_{322}|\leqslant\text{const.}\lim_{\varepsilon\to 0}\frac{1}{2^{d+2}}\int\left|J\left(\frac{t+s}{2},\frac{x+y}{2}\right)\right|H^{\varepsilon}\left(\frac{t-s}{2}\right)$$

$$\times\left|\gamma\cdot(\nabla H^{\varepsilon}_{d})\left(\frac{x-y}{2}\right)(x-y)\cdot\nabla\lambda(x)\right|$$

$$\times|\rho(s,y)-\rho(t,x)|\,\pi_{t,x}(dk)\,dy\,ds\,dx\,dt=0$$

because $z\gamma\cdot\nabla H^{\varepsilon}_{d}(z/2)=\varepsilon^{-d}G(z/\varepsilon)$ for some compactly supported smooth function G and $\rho(t,x)-\rho(s,y)$ vanishes as (t,x) approaches (s,y). Hence

$$\Omega_{32}=-\lim_{\varepsilon\to 0}\frac{1}{2^{d+1}}\gamma\cdot\int\nabla_{y}[J^{\varepsilon}(t,x,s,y)\,q(k;\rho(t,x))](\lambda(x)-\lambda(y))$$

$$\times\pi_{x,t}(dk)\,dy\,ds\,dx\,dt$$

$$+\lim_{\varepsilon\to 0}\frac{1}{2^{d+2}}\gamma\cdot\int(\nabla J)\left(\frac{t+s}{2},\frac{x+y}{2}\right)H^{\varepsilon}\left(\frac{t-s}{2}\right)H^{\varepsilon}_{d}\left(\frac{x-y}{2}\right)$$

$$\times(\lambda(x)-\lambda(y))\,q(k;\rho(t,x))\,\pi_{t,x}(dk)\,dy\,ds\,dx\,dt$$

$$=-\lim_{\varepsilon\to 0}\frac{1}{2^{d+1}}\gamma\cdot\int\nabla_{y}[J^{\varepsilon}(t,x,s,y)\,q(k;\rho(s,y))](\lambda(x)-\lambda(y))$$

$$\times\pi_{t,x}(dk)\,dy\,ds\,dx\,dt \tag{6.10}$$

because $\lambda(x)-\lambda(y)$ vanishes as x approaches y. From (6.8) we deduce

$$\Omega_1+\Omega_2=\lim_{\varepsilon\to 0}\frac{1}{2^{d+1}}\int J^\varepsilon(t,x,s,y)\,q(k,\rho(t,x))\gamma\cdot\nabla\lambda(y)\,\pi_{t,x}(dk)\,dy\,ds\,dx\,dt$$

This and (6.10) imply

$$\Omega_1+\Omega_2+\Omega_3=-\lim_{\varepsilon\to 0}\frac{1}{2^{d+1}}\gamma\cdot\int\nabla_y[J^\varepsilon(t,x,s,y)(\lambda(x)-\lambda(y))\,q(k;\rho(t,x))]$$
$$\times\pi_{t,x}(dk)\,dy\,ds\,dx\,dt$$

If we integrate with respect to y first, we evidently get zero. The result follows.

From this result, there follows

Corollary 6.3. $\int|k-\rho(t,x)|\,\pi_{t,x}(dk)\,dx$ is nonincreasing in t, for $t>0$.

Proof. First, we assume that ρ and π has bounded support in the space variable. (Note that if it does so initially, the same is true for all time. In (6.6), we choose a test function $J(t,x)$ which is constant within the spatial support of ρ for each t. In this case, the term in (6.4) with $\nabla_x J$ is identically zero, and so (6.6) reduces to

$$\int\partial_t J(t,x)\,|k-\rho(t,x)|\,\pi_{t,x}(dk)\,dx\,dt\geqslant 0$$

letting J approximate $\mathbb{1}(t\in[T_1,T_2])$, we get

$$\int\{|k-\rho(T_1,x)|-|k-\rho(T_2,x)|\}\,\pi_{t,x}(dk)\,dx\geqslant 0$$

in other words, the integral of $|k-\rho|$ is nonincreasing in t.

For general ρ, we choose a sequence ρ_k of compactly supported functions which are equal to ρ on $[0,T]\times[-k,k]^d$. Similarly, we set up a sequence of processes $\eta_t^{(k)}$ with initial density profile $\rho_k(0,\cdot)$. Then we couple $\eta^{(k)}$ with η, and note that for some fixed positive time, with probability 1, no discrepancies between the two enter into $[-N(k-1), N(k-1)]^d$. Similarly, for some positive time, no discrepancies between ρ and ρ_k enter into $[-k+1,k-1]^d$.

This tells us that, for some positive time, the integral of $|k-\rho|$ is nonincreasing in t on $[-k+1,k-1]^d$. But we can cover all on $\mathbb{R}^d$ by shifted versions of this box, and so the same is true on $\mathbb{R}^d$ for some fixed positive time. Finally, if it is true for some positive time, we can treat a later profile as a new initial profile and start over, so that the

Finally, we come to the

Proof of Theorem 1.2. We can find a smooth function ρ_0^ε with $\int |\rho_0(x) - \rho_0^\varepsilon(x)|\, dx < \varepsilon$. Let ρ^ε be the distribution solution of (1.3) with initial condition ρ_0^ε. Then, using Corollary 6.3,

$$\int |k - \rho^\varepsilon(t, x)|\, \pi_{t,x}(dk)\, dx < \varepsilon$$

for almost all $t > 0$. Since ρ^ε converges to ρ in L^1-sense, we conclude

$$\begin{aligned} \lim_{t \to 0} \int |k - \rho_0(x)|\, \pi_{t,x}(dk)\, dx &= \lim_{t \to 0} \int |k - \rho(t, x)|\, \pi_{t,x}(dk)\, dx \\ &= \lim_{t \to 0} \lim_{\varepsilon \to 0} \int |k - \rho^\varepsilon(t, x)|\, \pi_{t,x}(dk)\, dx = 0 \end{aligned}$$

This and Corollary 6.3 imply that $\pi_{t,x} = \delta_{\rho(t,x)}$ for all t, x.

Theorem 1.2 is hereby proved.

7. APPLICATION TO TASEP WITH BLOCKAGE

In this section, we prove a result that allows us to obtain a partial answer to the question of Janowsky and Lebowitz [JL2]: What is the threshold $\bar{\lambda}$ above which a lowered jump rate at the origin does not disturb the hydrodynamic limit for the system?

In fact, we will do more than this, finding a hydrodynamic limit for the process with continuous but nondifferentiale λ, and obtaining nontrivial upper and lower bounds for the case with λ piecewise continuous. We will work in one dimension, with the assumption that $p(-1) = 1$, that is, the particles jump only one site to the left. This assumption is essential for the results of this section. It allows us to compare the particle densities for processes with different jump rates.

We shall begin by assuming that our initial data $\rho_0 \in L^1 \cap L^\infty$. As before, let $\rho(t, x)$ be the unique solution of (1.3), (1.4); then define $r(t, x) = \int_{-\infty}^x \rho(t, y)\, dy$. In this case,

$$\partial_t r(t, x) - \lambda(x)\, h(\partial_x r) = 0 \tag{7.1}$$

If ρ is the entropy solution of (1.3), we say that the corresponding r is the *viscosity solution* of (7.1).

This function r will give the macroscropic densities for the process $\zeta_t(u)=\sum_{v=-\infty}^{u}\eta_t(v)$, where η_t is a PdM under the assumptions described above. Let r_0 be its initial data, which have $r_0(x)=\int_{-\infty}^{x}\rho_0(y)\,dy$.

By coupling a process with jump rate λ_1 to one with rate λ_2, where $\lambda_1\leqslant\lambda_2$ everywhere, with each process having the same initial configuration, we find that the coupled process always has particles in its sub-process farther to the left than the corresponding particles in the sub-process with λ_1. In other words, letting $\zeta^{(1)}$ and r_1 corresponding to the λ_1 process as described above and $\zeta^{(2)}$ and r_2 to the process with λ_2, we find that $\zeta^{(1)}(u)\leqslant\zeta^{(2)}(u)$ almost surely, and thus that $r_1(t,x)\leqslant r_2(t,x)$.

We also make the assumption that the flux function h is stictly concave. We are primarily interested in the SEP case, for which $h(\rho)=\rho(1-\rho)$.

For a PDE in the form of (7.1), we may use the Hopf–Lax Formula [ES], a standard result in PDE theory:

Lemma 7.1 (Hopf–Lax Formula). Let r be the viscosity solution of (7.1); then

$$r(t,x)=\inf_{w\in\mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda(w(s)),\omega'(s))\,ds\right\} \tag{7.2}$$

where $L(\lambda,q)$ is the Lagrangian defined by $L(\lambda,q)=\sup_p\{pq+\lambda h(p)\}$, and where $\mathscr{W}_{t,x}=\{w\in C^1[0,T]\colon w(t)=x\}$.

Note that $-h$ is convex and positive, and that L is nondecreasing in λ. Moreover, since $h(0)=0$, we clearly have $L\geqslant 0$.

For the process with λ piecewise continuous and equal to its right or left limit at each discontinuity, it is sufficient to consider λ with only one discontinuity, occurring at the origin. Denote the upper semicontinuous version by λ^+ and the lower semicontinuous by λ^-.

We may now choose a sequence of smooth functions $\lambda_n^+\downarrow\lambda^+$ pointwise. Let r_n^+ be the viscosity solution of (7.1) with λ_n^+; then r_n^+ is a nonincreasing sequence of functions.

Similarly, we choose a sequence of smooth functions λ_n^- increasing to λ^-. We let r_n^- be the viscosity solution of (7.1) with λ_n^-, and find that the sequence r_n^- is increasing.

Each r_n^+ is clearly an upper limit for the hydrodynamic limit of the process with jump rate λ, and each r_n^- is similarly a lower limit. If we can find conditions in which these sequences converge to the same function, we will have found a hydrodynamic limit for the process with λ. We now pursue this approach to the problem.

Lemma 7.2. Let

$$r^+(t,x)=\inf_{w\in \mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda^+(w(s)),w'(s))\,ds\right\}$$

Then

$$\lim_{n\to\infty} r_n^+(t,x)=r^+(t,x)$$

Proof. We note first that, since $\lambda^+\leqslant\lambda_n^+$ and since L is nondecreasing in its λ variable,

$$\begin{aligned}r(t,x)&=\inf_{w\in \mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda^+(w(s)),\omega'(s))\,ds\right\}\\ &\leqslant\inf_{w\in \mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda_n^+(w(s)),\omega'(s))\,ds\right\}=r_n^+(t,x)\end{aligned}$$

On the other hand, for any $w\in\mathscr{W}_{r,x}$,

$$r_n^+(t,x)\leqslant r_0(w(0))+\int_0^t L(\lambda_n^+(w(s)),w'(s))\,ds$$

and so

$$\lim_{n\to\infty} r_n^+(t,x)\leqslant r_0(w(0))+\lim_{n\to\infty}\int_0^t L(\lambda_n^+(w(s)),w'(s))\,ds$$

We may pass to the limit in the right side by bounded convergence, which tells us that

$$\lim_{n\to\infty} r_n^+(t,x)\leqslant r_0(w(0))+\int_0^t L(\lambda^+(w(s)),w'(s))\,ds$$

But since this is true for any w,

$$\lim_{n\to\infty} r_n^+(t,x)\leqslant \inf_{w\in \mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda^+(w(s)),w'(s))\,ds\right\}$$

which equals $r^+(t,x)$ by definition.

There is also a corresponding result for the function r_n^-:

Lemma 7.3. Let

$$r^-(t,x)=\inf_{w\in\mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda^-(w(s)),w'(s))\,ds\right\}$$

Then

$$\lim_{n\to\infty} r_n^-(t,x)=r^-(t,x)$$

Proof. Again, since $\lambda^-\geqslant\lambda_n^-$ and since L is increasing in its λ variable,

$$\begin{aligned}r(t,x)&=\inf_{w\in\mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda^+(w(s)),\omega'(s))\,ds\right\}\\&\geqslant\inf_{w\in\mathscr{W}_{t,x}}\left\{r_0(w(0))+\int_0^t L(\lambda_n^+(w(s)),\omega'(s))\,ds\right\}=r_n^-(t,x)\end{aligned}$$

We now concentrate on the opposite inequality. Let $W^{1,1}$ denote the space of functions on $[0,T]$ that are weakly differentiable with an integrable derivative. For this, we recall a standard fact from analysis:

Lemma 7.4. Given a positive function ϕ with $\phi(q)/|q|\to\infty$, if a sequence of functions $\{w_n\}$ has $\int_0^t\phi(w_n'(s))\,ds$ uniformly bounded, then $\{w_n\}$ is equicontinuous in $[0,t]$.

The proof of the lemma will be given later.

For any n, we choose a function w_n with

$$r_n^-(t,x)+\frac{1}{n}\geqslant r_0(w_n(0))+\int_0^t L(\lambda_n^-(w_n(s)),w_n'(s))\,ds$$

Then the sequence $\{w_n\}$ satisfies the conditions for equicontinuity in Lemma 7.4, using $\phi(q)=L(c_0,q)$ where c_0 is a lower limit for the range of λ. Since $\{w_n\}$ is therefore equicontinuous, we can find a uniform limit w of some subsequence $\{w_{k_n}\}$. Since the r_n is nondecreasing, it has a unique limit point, and so it suffices to find the limit of r_{k_n}. Because of this, we may henceforth treat $\{w_n\}$ as converging uniformly to w. Moreover, the sequence $\int_0^t\phi(w')\,ds$ is uniformly bounded. Since the function ϕ grows faster than the linear function, the sequence w_n' is uniformly integrable. Hence the sequence w_n' converges weakly to an integrable function v. This implies that w is weakly differentiable and $v=w'$.

Also, for a fixed function $g(s)$, it will show that

$$\liminf_{n\to\infty} \int_0^t L(g(s), w_n'(s))\, ds \geqslant \int_0^t L(g(s), w'(s))\, ds$$

To see this, recall the definition of L. Since w_n' converges weakly to w', for any bounded measurable function p,

$$\liminf_n \int_0^t L(g, w_n')\, ds \geqslant \liminf_n \int_0^t [p(s)\, w_n'(s) + g(s)\, h(p(s))]\, ds$$

$$= \int_0^t [p(s)\, w'(s) + g(s)\, h(p(s))]\, ds$$

We not set $p(s) = p_k(s) = P_k(g(s), w'(s))$ where $P_k(g, q)$ denotes the unique maximizer of

$$\sup_{p:\, |p| \leqslant k} (pq + gh(p))$$

(Uniqueness follows from the strict concavity of h.) Hence by the Monotone Convergence Theorem,

$$\liminf_n \int_0^t L(g, w_n')\, ds \geqslant \lim_k \int_0^t [p_k w' + gh(p_k)]\, ds$$

$$= \int_0^t L(g, w_n')\, ds$$

We now proceed with the problem. Since $w_n \to w$ uniformly, each λ_n is continuous, and $\lambda_n^- \uparrow \lambda^-$, for fixed k and large n we have $\lambda_n^-(w_n) \geqslant \lambda_k^-(w) - 1/k$ everywhere. Therefore, for any k, we have

$$\liminf_{n\to\infty} \int_0^t L(\lambda_n^-(w_n(s))\, w_n'(s))\, ds \geqslant \liminf_{n\to\infty} \int_0^t L\left(\lambda_k^-(w(s)) - \frac{1}{k}, w_n'(s)\right) ds$$

$$\geqslant \int_0^t L\left(\lambda_k^-(w(s)) - \frac{1}{k}, w'(s)\right) ds$$

(by lower semicontinuity);

$$\liminf_{n\to\infty} \int_0^t L(\lambda_n^-(w_n(s))\, w_n'(s))\, ds \geqslant \liminf_{k\to\infty} \int_0^t L\left(\lambda_k^-(w(s)) - \frac{1}{k}, w_n'(s)\right) ds$$

$$\geqslant \int_0^t L\left(\lambda_k^-(w(s)) - \frac{1}{k}, w'(s)\right) ds$$

(by Fatou's Lemma)

$$= \int_0^t L(\lambda^-(w(s)), w'(s))$$

Now, it is also true by uniform convergence that $r_0(w_n(0)) \to r_0(w(0))$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} r_n^-(t,x) &\geqslant \liminf_{n\to\infty} r_0(w_n(0)) + \int_0^t L(\lambda_n^-(w_n(s)), w_n'(s))\,ds \\
&\geqslant \liminf_{n\to\infty} r_0(w(0)) + \int_0^t L(\lambda^-(w(s)), w'(s)) \\
&\geqslant \inf_{w\in\mathscr{W}_x}\left\{r_0(w(0)) + \int_0^t L(\lambda^-(w(s)), w'(s))\,ds\right\} \\
&= r^-(t,x)
\end{aligned}$$

The result follows.
We continue with the proof of Lemmas 7.4.

Proof of Lemma 7.4. Without loss of generality, we may assume $\phi(q) = |q|\, j(q)$ where $j(q_1) \leqslant j(q_2)$ if $|q_1| \leqslant |q_2|$, and $\lim j(q) = +\infty$ as $|q|$ goes to infty. Suppose $t_2 \geqslant t_1$. We then have

$$\begin{aligned}
|w_n(t_2) - w_n(t_1)| = \left|\int_{t_1}^{t_2} w'(s)\,ds\right| &\leqslant l(t_2 - t_1) + \int_0^t |w'(s)|\ \mathbb{1}(|w'(s)| \geqslant l)\,ds \\
&= l(t_2 - t_1) + \left|\int_0^t \frac{\phi(w'(s))}{j(w_n'(s))}\mathbb{1}(|w'(s)| \leqslant l)\,ds\right| \\
&\leqslant l(t_2 - t_1) + j(l)^{-1}\int_0^t \phi(w')\,ds
\end{aligned}$$

for every positive l. If $\Gamma(t_2 - t_1)$ denotes the infimum of the right-hand side over l, it is not hard to show that $\lim_{\delta\to 0}\Gamma(\delta) = 0$. This evidently completes the proof. ∎

From Lemmas 7.2 and 7.3 we obtain the following corollaries, which give us information about the density profiles in certain cases:

Corollary 7.5. Suppose λ is piecewise continuous; then the measure $\pi_{t,x}$ is concentrated on the interval $[r^-(t,x), r^+(t,x)]$ R-almost surely.

Proof. By coupling with the processes with speed λ_n^-, we find that the expected values of ζ are at least $r_n^-(t, x)$, and thus, passing to the limit, at least $r^-(t, x)$. Similarly, by comparison with the processes with rate λ_n^+, the expected values can be no more than $r^+(t, x)$. The result follows.

Corollary 7.6. Suppose λ is continuous; then $\pi_{t,x}(dk) = \delta_{r(t,x)}$ R-almost surely (see the discussion at the beginning of Chapter 6 for π and R; here we define π using the summation process ζ instead of η as before), where $r(t, x)$ is as defined by (7.2).

Proof. If λ is continuous, then the definitions of r^+ and r^- coincide, and by Corollary 7.5, the result follows. ▮

We next return our attention to the question concerning the blockage at a single point. We will approximate this case using piecewise constant functions λ_n. We do not have an exact hydrodynamic limit in this case. However, we make use of the fact that the convexity of $L(\lambda, q)$ in q for fixed λ allows us to simplify the Hopf's Formula expression when λ is piecewise constant. In this case, the infimum in Hopf's Formula may be taken over piecewise linear functions w which are linear over each interval with λ constant. This is true because the convexity of L tells us that if $\lambda(w(s))$ is constant for s in some $[t_1, t_2]$, then, by Jensen's inequality, the integral of $L(\lambda(w(s)), w'(s))$ over that interval is greater than or equal to $(t_2 - t_1)\, L(\lambda, (w(t_2) - w(t_1))/(t_2 - t_1))$.

The results of this section are also true with the condition $\rho_0 \in L^1$ dropped. To see this, choose a bounded measurable ρ_0 and compare it by coupling with truncated versions of itself; since the speed of propagation of particles is finite, we have arbitrarily large regions over which there are no discrepancies between the coupled processes, and the hydrodynamic limit for the truncated version will be the limit for the non-truncated version also. (For $\rho_0 \notin L^1$, it is necessary to define r differently; we will use $r(t, x) = \int_0^x \rho(t, x)$. This does not mterially affect the relationship between ρ and r in the sense that, if ρ is an entropy solution of (1.3), r is still a viscosity solution of (7.1).)

This allows us to obtain our lower bound for the threshhold $\bar{\lambda}$ of blockages that do not affect the hydrodynamics of the process with blockage at the origin. Consider the process with constant initial density ρ_0; let $r_0(x) = x\rho_0$. Let r be the entropy solution of (7.1) with initial data r_0. Let $\pi_{t,x}$ be as in the previous Corollaries, for the process with rate 1 away from the origin and λ_0 at 0. Let $r(t, x)$ be the entropy solution of (7.1) with initial data r_0 and $\lambda \equiv 1$. Then we have:

Theorem 7.7. Assume $h(\rho) = \rho(1 - \rho)$ in (7.1) and suppose $\lambda_0 \geqslant 4h(\rho_0)$; then $\pi_{t,x} = \delta_{r(t,x)}$, R-almost surely.

Proof. Let λ_n be a sequence of continuous functions increasing pointwise to λ, where $\lambda(0)=\lambda_0$ and $\lambda\equiv 1$ elsewhere. Let $r_n(t,x)$ be the entropy solution of the PDE (7.1) with λ_n. Then the reasoning of Lemma 7.3, followed exactly, tells us that

$$\lim_{n\to\infty} r_n(t,x) = \inf_{w\in \mathscr{W}_{t,x}} \left\{ r_0(w(0)) + \int_0^t L(\lambda(w(s)), w'(s))\, ds \right\} \quad \blacksquare$$

Now, the convexity of L for fixed λ tells us that it is sufficient to consider the infimum over the set $\tilde{\mathscr{W}}_{t,x}$ of piecewise linear functions on $[0,t]$ with terminal point x. This infimum is actually a minimum, since L increases quadratically in its second variable and everything else involved only increases or decreases linearly, and so large values of w' only increase $r_0(w(0)) + \int_0^t L(\lambda(w(s)), w'(s))\, ds$.

What we need to prove is that the minimizing $w\in\tilde{\mathscr{W}}_x$ spends no time at the origin. Suppose $w(s)=0$ for some s, and let t_0 be the last time for which this is true. Then w also minimizes $r_0(w(0)) + \int_0^{t_0} L(\lambda(w(s)), w'(s))\, ds$ among piecewise linear functions on $[0,t_0]$ with terminal point 0. Hence without loss of generality we may assume $t=t_0$.

What we are trying to show, then, is that the minimizer of

$$r_0(-(t-s)z) + (t-s)\, L(1,z) + sL(\lambda_0, 0)$$

over z and s (where z represents the slope of w away from the origin and s represents the time spent at 0) has $s=0$. Set $k(q)=L(1,q)$. Then $L(\lambda,q)=\lambda k(q/\lambda)$ and $r_0(x)=\rho_0 x$, the expression to be minimized equals

$$\begin{aligned} &-\rho_0 z(t-s) + (t-s)\, k(z) + s\lambda_0 k(0) \\ &\quad = (t-s)(-\rho_0 z + k(z)) + s\lambda_0 k(0) \end{aligned}$$

In the SEP, $h(p)=p(1-p)$, and $k(q)=\frac{1}{4}(1+q)^2$. Thus we are actually minimizing

$$(t-s)\left(-\rho_0 z + \frac{1}{4}(1+z)^2\right) + \frac{s}{4}\lambda_0 \tag{7.3}$$

over z and s. We first minimize over z for fixed s:

$$\frac{d}{dz}\left\{(t-s)\left(-\rho_0 z + \frac{1}{4}(1+z)^2\right) + \frac{s}{4}\lambda_0\right\} = (t-s)\left(-\rho_0 + \frac{1}{2}(1+z)\right)$$

either $s=t$ or $(1+z)=2\rho_0$; $z=-1+2\rho_0$.

Assume $s \neq t$. In this case, $k(z) = \frac{1}{4}(1-(1-2\rho_0))^2 = \rho_0^2$. (7.3) then reduces to

$$(t-s)(\rho_0(1-2\rho_0)+\rho_0^2)+\frac{s}{4}\lambda_0 = (t-s)\,h(\rho_0)+\frac{s}{4}\lambda_0$$

Clearly the case $t = s$ leads to the same minimum value. Finally, we minimize oer s, and discover that it has a minimum at $s = 0$ because $\lambda_0 > 4h(\rho_0)$.

From this we conclude that the Hopf's Formula minimizer w is a straight line. (It spends no time at zero, and by convexity it must have the same slopes on the left and right if it crosses the origin.) In order words,

$$\lim_{n \to \infty} r_n(t, x) = \inf_z r_0(x - tz) + tL(1, z)$$

But this is also the expression for the solution $r^{(1)}$ of the PDE with $\lambda \equiv 1$. Since $r_n(t, x) \to r^{(1)}(t, x)$ pointwise, and since the process with blockage λ_0 at 0 is trapped between the two, we conclude that its particle densities are the same as if there were no blockage at all. This completes the proof of Theorem 7.7.

Here, then, is a upper bound for the Janowsky-Lebowitz threshhold. It should be noted that there is still room for improvement in the study of this question; the simulations in [JL2] showed that values of λ_0 lower than this can still avoid disturbing the hydrodynamics of the process. In fact, the threshold value obtained in [JL2] is strictly less than $4h(\rho_0)$ for every $\rho_0 \in (0, 1)$.

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